Limits & Derivatives Worksheet SOLUTIONS
Math 1100-005
01/26/06

1. Find the limit (if it exists):

   (a) $\lim_{t\to 3} \frac{t^2+1}{t}$

$$\begin{aligned} \lim_{t\to 3} \frac{t^2+1}{t} &= \frac{3^2+1}{3} \\ &= \frac{9+1}{3} \\ &= \frac{10}{3} \end{aligned}$$

   (b) $\lim_{x\to \frac{1}{2}} \frac{2x-1}{6x-3}$

$$\begin{aligned} \lim_{x\to \frac{1}{2}} \frac{2x-1}{6x-3} &= \lim_{x\to \frac{1}{2}} \frac{2x-1}{3(2x-1)} \\ &= \lim_{x\to \frac{1}{2}} \frac{1}{3} \\ &= \frac{1}{3} \end{aligned}$$

   (c) $\lim_{x\to 0} \frac{\frac{1}{x-2}-1}{x}$

$$\begin{aligned} \lim_{x\to 0} \frac{\frac{1}{x-2}-1}{x} &= \lim_{x\to 0} \frac{\frac{1}{x-2}-\frac{x-2}{x-2}}{x} \\ &= \lim_{x\to 0} \frac{\frac{1-(x-2)}{x-2}}{x} \\ &= \lim_{x\to 0} \frac{1-x+2}{x(x-2)} \\ &= \lim_{x\to 0} \frac{3-x}{x(x-2)} \\ &= \text{undefined} \end{aligned}$$

2. Describe the intervals on which the function is continuous:

   (a) $f(x) = \frac{x+1}{2x+2}$
   This function is discontinuous at $x = -1$ because $x = -1$ gives us 0 in the denominator.
   So, it is continuous on the intervals $(-\infty, -1)$ and $(-1, \infty)$

   (b) $f(x) = \frac{1}{x^2+x-2} = \frac{1}{(x+2)(x-1)}$
   This function is discontinuous at $x = 1$ & $x = -2$ since then we get a 0 in the denominator.
   So, it is continuous on the intervals $(-\infty, -2)$ and $(-2, 1)$ and $(1, \infty)$

3. Find the slope of the tangent line at the given point:

(a) $f(x) = (x-1)^2$ at $(-2, 9)$

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{((-2+\Delta x)-1)^2 - (-2-1)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(-3+\Delta x)^2 - (-3)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(-3)^2 - 6\Delta x + (\Delta x)^2 - 9}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{9 - 6\Delta x + (\Delta x)^2 - 9}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-6\Delta x + (\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x(-6+\Delta x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} -6 + \Delta x \\
&= -6
\end{aligned}$$

4. Find the derivative using the definition of a derivative

(a) $f(x) = x^2 + 3$

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + 3 - (x^2+3)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 3 - x^2 - 3}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x(2x+\Delta x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} 2x + \Delta x \\
&= 2x
\end{aligned}$$

(b) $f(x) = 2x + 5$

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{2(x+\Delta x) + 5 - (2x+5)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x + 2\Delta x + 5 - 2x - 5}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} 2 \\
&= 2
\end{aligned}$$

5. Find the derivative:

(a) $f(x) = 3x^2 - x + \frac{1}{x} = 3x^2 - x + x^{-1}$

$$\begin{aligned} f'(x) &= 3(2)x^{2-1} - 1 + (-1)x^{-1-1} \\ &= 6x - 1 - x^{-2} \\ &= 6x - 1 - \frac{1}{x^2} \end{aligned}$$

(b) $f(x) = x^{\frac{1}{2}} + x^3 - 6$

$$\begin{aligned} f'(x) &= \frac{1}{2}x^{\frac{1}{2}-1} + 3x^{3-1} \\ &= \frac{1}{2}x^{\frac{-1}{2}} + 3x^2 \\ &= \frac{1}{2x^{\frac{1}{2}}} + 3x^2 \\ &= \frac{1}{2\sqrt{x}} + 3x^2 \end{aligned}$$

(c) $f(x) = \frac{2}{x^{\frac{5}{3}}} = 2x^{\frac{-5}{3}}$

$$\begin{aligned} f'(x) &= 2\left(\frac{-5}{3}\right)x^{\frac{-5}{3}-1} \\ &= \frac{-10}{3}x^{\frac{-8}{3}} \\ &= \frac{-10}{3x^{\frac{8}{3}}} \end{aligned}$$

(d) $f(x) = (x+1)(x^3 - 2x - 1)$ (USE THE PRODUCT RULE)

$$\begin{aligned} f'(x) &= (1)(x^3 - 2x - 1) + (x+1)(3x^2 - 2) \\ &= x^3 - 2x - 1 + (3x^3 + 3x^2 - 2x - 2) \\ &= 4x^3 + 3x^2 - 4x - 3 \end{aligned}$$

(e) $f(x) = \sqrt{x}(x^2 - x)$ (USE THE PRODUCT RULE)

$$\begin{aligned} f'(x) &= \frac{1}{2}x^{\frac{-1}{2}}(x^2 - x) + \sqrt{x}(2x - 1) \\ &= \frac{1}{2\sqrt{x}}(x^2 - x) + \sqrt{x}(2x - 1) \end{aligned}$$

(f) $f(x) = \frac{4x+2}{x-1}$ (USE THE QUOTIENT RULE)

$$\begin{aligned} f'(x) &= \frac{(4)(x-1) - (4x+2)(1)}{(x-1)^2} \\ &= \frac{4x - 4 - 4x - 2}{(x-1)^2} \\ &= \frac{-6}{(x-1)^2} \end{aligned}$$

(g) $f(x) = \frac{4x^2-3x}{x^{\frac{2}{3}}-x}$ (USE THE QUOTIENT RULE)

$$f'(x) = \frac{(8x-3)(x^{\frac{2}{3}}-x)-(4x^2-3x)(\frac{2}{3}x^{\frac{-1}{3}}-1)}{(x^{\frac{2}{3}}-x)^2}$$

6. The height h (feet) at time t (seconds) of a ball dropped off a building is given by: $h(t) = -16t^2 + 150$

   (a) Find the average velocity on the interval [1,2].

$$\begin{aligned} \frac{\Delta h}{\Delta t} &= \frac{h(2)-h(1)}{2-1} \\ &= \frac{-16(2)^2+150-(-16(1)^2+150}{1} \\ &= -64+150+16-150 \\ &= -48ft/s \end{aligned}$$

   (b) Find the instantaneous velocities when t=1 & t=2.

$$\begin{aligned} h'(t) &= -32t \\ h'(1) &= -32(1) = -32ft/s \\ h'(2) &= -32(2) = -64ft/s \end{aligned}$$

7. The revenue (dollars) of selling x units of calculators is given by: $R(x) = 50x - 0.5x^2$

   (a) Find the additional revenue when sales increase from 9 to 10.

$$\begin{aligned} \text{Additional Revenue} &= R(10)-R(9) \\ &= 50(10)-0.5(10)^2-(50(9)-0.5(9)^2) \\ &= 450-409.5 \\ &= \$40.5/unit \end{aligned}$$

   (b) Find the marginal revenue when x=10.

$$\begin{aligned} R'(t) &= 50-x \\ R'(10) &= 50-10 \\ &= \$40/unit \end{aligned}$$