Solutions to Review Problems for Exam 3

(1) a) $f(x) = e^{-\frac{x^2}{2}}$

$f'(x) = e^{-\frac{x^2}{2}}\left(-\frac{x^2}{2}\right)' = -xe^{-\frac{x^2}{2}}$ (Chain Rule)

b) $f(t) = \ln(t^2)e^t = 2\ln t\, e^t$

$f'(t) = \frac{2}{t}e^t + 2\ln t\, e^t$ (product rule)

c) $f(x) = 2^x \log_2 x$

$f'(x) = (2^x)'\log_2 x + 2^x(\log_2 x)'$

$= \ln 2 \cdot 2^x \log_2 x + 2^x \frac{1}{(\ln a) x}$

(since $2^x = e^{x\ln 2}$ and $\log_2 x = \frac{\ln x}{\ln 2}$)

(2) $f(x) = e^{-x^2}$

a. $f'(x) = e^{-x^2}(-x^2)' = -2xe^{-x^2}$

$f'(x) = 0 \Rightarrow x = 0$ since $e^{-x^2} \neq 0$

$(-\infty, 0)$ $x < 0 \Rightarrow f'(x) = -2xe^{-x^2} > 0$ increasing

$(0, \infty)$, $x > 0 \Rightarrow f'(x) = -2xe^{-x^2} < 0$ decreasing

b) Only critical value is 0; by 1st derivative test 0 is a relative maximum value. $f(0) = e^0 = 1$

relative maximum point is $(0,1)$.

No relative minimum points.

(3) $y e^x = y^2 + x - 2$

a) $y' e^x + y e^x = 2yy' + 1$

$$y' e^x - 2yy' = 1 - ye^x$$

$$y'(e^x - 2y) = 1 - ye^x$$

$$\boxed{y' = \frac{1 - ye^x}{e^x - 2y}}$$

b) Slope m of the tangent line at $(0,2)$ is

$$\frac{dy}{dx} = \frac{1 - 2 \cdot e^0}{e^0 - 2 \cdot 2} = \frac{1-2}{1-4} = \frac{-1}{-3} = \frac{1}{3}$$

$\boxed{y - 2 = \frac{1}{3}(x - 0)}$ or $\boxed{y = \frac{1}{3}x + 2}$

(4) $P(x) = R(x) - C(x) = 400x - \frac{x^2}{20} - (5000 + 70x)$

$\Rightarrow \quad P(x) = -\frac{x^2}{20} + 400x - 5000 - 70x$

$P(x) = -\frac{x^2}{20} + 330x - 5000.$

Find $\frac{dP}{dt}$ when $x = 200$ and $\frac{dx}{dt} = 5$

$\frac{dP}{dt} = -\frac{2x}{20}\frac{dx}{dt} + 330\frac{dx}{dt}$

$= -\frac{2\cdot 200}{20}\cdot 5 + 330\cdot 5$

$\frac{dP}{dt} = -100 + 1650 = \boxed{\$1550/\text{month.}}$

(5) $2p^2q = 10{,}000 + 9{,}000p^2$

Find $\frac{dq}{dp}$ by Implicit differentiation:

$\cancel{2}4pq + \cancel{2}p^2\frac{dq}{dp} = \cancel{2}\cdot 9{,}000p$

$p^{\cancel{2}}\frac{dq}{dp} = 9{,}000p - 2pq = \cancel{p}(9{,}000 - 2q)$

$p\frac{dq}{dp} = 9{,}000 - 2q$

Multiply both sides by $-\frac{1}{q}$!

$$\eta = -\frac{p}{q}\frac{dq}{dp} = -\frac{9{,}000-2q}{q}$$

When $p = 50$ and $q = 4502$ we get

$$\eta = -\frac{9{,}000 - 9004}{4502} = \boxed{\frac{4}{4502}}$$

The demand is inelastic $(\eta < 1)$, therefore the revenue will increase, with an increase in price.

(6) a. $\int x^2 - \frac{1}{x^2} + \frac{1}{\sqrt{x}}\, dx = \int x^2 dx - \int \frac{1}{x^2} dx + \int \frac{1}{\sqrt{x}} dx$

$$= \frac{x^3}{3} - \int x^{-2} dx + \int x^{-\frac{1}{2}} dx$$

$$= \frac{x^3}{3} - \frac{x^{-1}}{-1} + \frac{x^{3/2}}{3/2} + C$$

$$= \boxed{\frac{x^3}{3} + \frac{1}{x} + \frac{2}{3}x^{3/2} + C}$$

b. $\int x\sqrt{x^2+1}\, dx = \int x(x^2+1)^{1/2} dx.$

Substitute $u = x^2 + 1$

$du = 2x\, dx$

$\frac{1}{2} du = x\, dx$

$$\int u^{1/2}\,du = \frac{u^{3/2}}{3/2}+C = \frac{2}{3}u^{3/2}+C$$

$$= \boxed{\frac{2}{3}(x^2+1)^{3/2}+C}$$

c. $\displaystyle\int \frac{x^2}{3x^3+7}\,dx$ Substitute $u = 3x^3+7$

$du = 9x^2\,dx$

$x^2\,dx = \frac{1}{9}\,du$

$$= \int \frac{1}{u}\cdot\frac{1}{9}\,du = \frac{1}{9}\int\frac{1}{u}\,du = \frac{1}{9}\ln|u|+C$$

$$= \boxed{\frac{1}{9}\ln|3x^3+7|+C}$$

7. $R(x) = \int R'(x)\,dx = \int 400-2x\,dx = \int 400\,dx - 2\int x\,dx$

$= 400x - 2\frac{x^2}{2}+C$

$= 400x - x^2 + C$

But $R(0) = 0$ always so

$R(0) = 0 = C \Rightarrow \boxed{R(x) = 400x - x^2}$

$C(x) = \int C'(x)\,dx = \int -x+100\,dx = -\int x\,dx + 100\int dx$

$= -\frac{x^2}{2} + 100x + C$

Fixed costs are 1,000 so $C(0) = 1{,}000$

$$C(0) = 1{,}000 = C$$

$$C(x) = -\frac{x^2}{2} + 100x + 1000.$$

$$P(x) = R(x) - C(x)$$

so $\boxed{P(x) = 400x - x^2 - \left(-\frac{x^2}{2} + 100x + 1{,}000\right)}$

or $P(x) = 400x - x^2 + \frac{x^2}{2} - 100x - 1000$

$$\boxed{P(x) = -\frac{x^2}{2} + 300x - 1000}$$

~~P is maximized wh~~

critical value: $P'(x) = -x + 300 = 0$

$$x = 300.$$

Since $P''(x) = -1 < 0$ all x, concave down

$\Rightarrow$ $\boxed{\text{Maximum profit occurs at } x = 300}$