$\boxed{\text{Solutions}}$

MATH 1100-001
EXAM III

Instructions:. There are 6 problems. Show your work for full credit. Put your name on the exam. Calculators and other electronic devices (including cell phones) are not allowed. You are allowed one side of an 8.5 by 11 inch sheet of notes.

1. (15) Find the derivatives of the following functions:

a. $f(x) = e^{(\frac{1}{x^2}+1)}$

b. $f(t) = \ln\sqrt{t+1}$

c. $f(x) = 10^{0.1x}$

a. $f'(x) = e^{(\frac{1}{x^2}+1)}\left(\frac{1}{x^2}+1\right)' = e^{(\frac{1}{x^2}+1)}(x^{-2}+1)' = \boxed{e^{(\frac{1}{x^2}+1)}(-2x^{-3})}$

or $\boxed{-\frac{2}{x^3}e^{(\frac{1}{x^2}+1)}}$

b. $\ln\sqrt{t+1} = \ln(t+1)^{1/2} = \frac{1}{2}\ln(t+1)$

$f'(t) = \frac{1}{2}\cdot\frac{1}{t+1}\cdot(t+1)' = \boxed{\frac{1}{2}\cdot\frac{1}{t+1}}$ or $\boxed{\frac{1}{2t+2}}$

or $\frac{1}{(t+1)^{1/2}}\left[(t+1)^{1/2}\right]' = \boxed{\frac{\frac{1}{2}(t+1)^{-\frac{1}{2}}}{(t+1)^{1/2}}} = \frac{\frac{1}{2}}{(t+1)}$

c. $f(x) = 10^{0.1x} = e^{0.1x\ln 10}$, $f'(x) = e^{0.1x\ln 10}(0.1x\ln 10)'$

$= \boxed{0.1\ln 10\, e^{0.1x\ln 10}}$

or $\boxed{0.1\ln 10\; 10^{0.1x}}$

2. (20 points) Consider the function $f(x) = 2xe^{-3x}$.

a. Find $f'(x)$.

b. Find the critical values, and the intervals on which f is increasing and decreasing.

c. Find any relative maximum and minimum points.

a. Product Rule: $f'(x) = 2e^{-3x} + 2x(e^{-3x})'$

$$= 2e^{-3x} + 2x\cdot(-3)e^{-3x}$$

$$\boxed{f'(x) = 2e^{-3x} - 6xe^{-3x}}$$

b. $f'(x) = 2e^{-3x} - 6xe^{-3x} = 0$

$$e^{-3x}(2-6x) = 0$$

$e^{-3x} > 0 \Rightarrow (2-6x) = 0 \Rightarrow 6x = 2,\ \boxed{x = \frac{1}{3}}$ critical value.

$x < \frac{1}{3}$, say $x = 0$, $f'(0) = 2e^{0} = 2 > 0$

$x > \frac{1}{3}$ say $x = 1$, $f'(1) = 2e^{-3} - 6e^{-3} = -4e^{-3} < 0$

So $\boxed{\begin{array}{l} f \text{ is increasing on } (-\infty, \frac{1}{3}) \\ f \text{ is decreasing on } (\frac{1}{3}, \infty) \end{array}}$

f': $+$ $-$

$\frac{1}{3}$

c. By the 1st derivative test and part b, f has a relative max point at $x = \frac{1}{3}$. $f(1/3) = \frac{2}{3}e^{-3\cdot\frac{1}{3}} = \frac{2}{3}e^{-1}$

Rel. Max Point: $\boxed{(\frac{1}{3}, \frac{2}{3}e^{-1})}$

No relative min. point.

3. (20 points) Consider the following equation:

$$y^2x - 4x^2y = -4$$

a. Use implicit differentiation to find $\frac{dy}{dx}$.

b. Find the equation of the tangent line at the point $(1, 2)$.

a. Using products and chain rules:

$$2y\frac{dy}{dx}\cdot x + y^2 - 8xy - 4x^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2yx - 4x^2) = 8xy - y^2$$

$$\boxed{\frac{dy}{dx} = \frac{8xy - y^2}{2yx - 4x^2}}$$

b. At the point $(1, 2)$ the slope is $\frac{dy}{dx}$ with $x = 1$, $y = 2$

$$m = \frac{8\cdot 1\cdot 2 - 2^2}{2\cdot 2 - 4} = \frac{12}{0} \quad \text{undefined}.$$

So, a vertical line through the point.

$$\boxed{x = 1}$$

4. (20 points) Suppose that the demand for a product is given by

$$p^2 + 4p + q = 200$$

where p is the price in dollars and q is the quantity demanded. Find the elasticity of demand when $p = \$10$. How would the revenue be affected by a price increase?

First find $\frac{dq}{dp}$ by implicit differentiation, since

$\eta = -\frac{p}{q}\frac{dq}{dp}$ (elasticity of demand). p - variable, q - function

$$2p + 4 + \frac{dq}{dp} = 0$$

$$\frac{dq}{dp} = -2p - 4 \qquad \text{Multiply by } -\frac{p}{q}$$

$$\Rightarrow \quad \eta = -\frac{p}{q}(-2p-4) \quad \Rightarrow \quad \boxed{\eta = \frac{p}{q}(2p+4)}$$

If $p = 10$, then solving for q:

$$10^2 + 4 \cdot 10 + q = 200$$

$$q = 200 - 150 - 40 = 60$$

$$\eta = \frac{10}{60}(20+4) = \frac{240}{60} = \boxed{4}$$

Since $\eta > 1$, the demand is elastic, and an increase in price will lead to a decrease in revenue.

5. (15 points) Compute the following indefinite integrals.

a.

$$\int 2x^3 - \frac{3}{\sqrt[3]{x}}\,dx$$

$$\int 2x^3 dx - \int \frac{3}{\sqrt[3]{x}}\,dx = 2\int x^3 dx - 3\int \frac{1}{x^{1/3}}\,dx$$

$$= 2\frac{x^4}{4} - 3\int x^{-1/3}\,dx = \frac{x^4}{2} - 3\cdot\frac{3}{2}x^{2/3} + C$$

$$= \boxed{\frac{x^4}{2} - \frac{9}{2}x^{2/3} + C}$$

b.

$$\int \frac{x}{\sqrt{x^2+4}}\,dx$$

$$\int \frac{x}{(x^2+4)^{1/2}}\,dx = \int x(x^2+4)^{-1/2}\,dx.$$

sub. $u = x^2+4$

$du = 2x\,dx$

$x\,dx = \frac{1}{2}\,du$

$$\int u^{-1/2}\frac{1}{2}\,du = \frac{1}{2}\int u^{-\frac{1}{2}}\,du = \frac{1}{2}\,\frac{u^{1/2}}{1/2} + C = u^{1/2} + C = \boxed{(x^2+4)^{1/2} + C}$$

or $\boxed{\sqrt{x^2+4} + C}$

c.

$$\int e^{-0.1x}\,dx$$

Sub. $u = -0.1x$

$du = -0.1\,dx$ or $dx = \frac{-1}{0.1}\,du = -10\,du$

$$\int e^u(-10)\,du = -10\int e^u\,du = -10e^u + C = \boxed{-10e^{-0.1x} + C}$$

6. (10 points) Suppose that the marginal cost function is given by

$$C'(x) = 10 - x$$

Also, suppose that the cost of producing 10 units is $1,000. Find the cost function.

$$C(x) = \int C'(x)\,dx = \int 10 - x\,dx$$

$$= \int 10\,dx - \int x\,dx$$

$$C(x) = 10x - \frac{x^2}{2} + C.$$

But $C(10) = 1{,}000 = 10 \cdot 10 - \frac{10^2}{2} + C$ Solve for C

$$1{,}000 = 100 - 50 + C$$

$$C = 950$$

$$\boxed{C(x) = 10x - \frac{x^2}{2} + 950}$$