Solutions

MATH 1100-001

EXAM I

Instructions:. Put your name on the exam. Show your work for full credit. Calculators and other electronic devices (including cell phones) are not allowed. You are allowed one side of an 8.5 by 11 inch sheet of notes.

1. (5 points each) For $f(x) = \frac{x^2-x-2}{x^2-1}$ evaluate the following limits:

a. $\lim_{x \to -1} f(x)$

Setting $x = -1 \Rightarrow \frac{(-1)^2+1-2}{1-1} = \frac{0}{0}$. Might have a limit. Factor

$$\lim_{x \to -1} \frac{\cancel{(x+1)}(x-2)}{\cancel{(x+1)}(x-1)} = \lim_{x \to -1} \frac{x-2}{x-1} = \frac{-1-2}{-1-1} = \frac{-3}{-2} = \frac{3}{2}$$

b. $\lim_{x \to 1} f(x)$

Setting $x = 1$: $\frac{1-1-2}{1-1} = -\frac{2}{0}$. The limit will not exist.

c. $\lim_{x \to 0} f(x)$

Since denominator is non-zero at $x=0$, we can substitute:

$$\lim_{x \to 0} \frac{x^2-x-2}{x^2-1} = \frac{0-0-2}{0-1} = \frac{-2}{-1} = 2$$

d. At what points is f(x) discontinuous? Only at points where denominator is zero: $x = 1, -1$.

2. (20 points) Using the definition of the derivative, find $f'(x)$ where $f(x) = x - 2x^2$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim \frac{x+h - 2(x+h)^2 - [x - 2x^2]}{h}$$

$$= \lim_{h\to 0} \frac{x + h - 2(x^2 + 2xh + h^2) - x + 2x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x} + h - \cancel{2x^2} - 4xh - 2h^2 - \cancel{x} + \cancel{2x^2}}{h}$$

$$= \lim_{h\to 0} \frac{h - 4xh - 2h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(1 - 4x - 2h)}{\cancel{h}}$$

$$= \lim_{h\to 0} 1 - 4x - h = 1 - 4x$$

3. (20 points) Let $f(x) = (2 + x)(x^2 - 1)$. Find the equation of the tangent line to the graph when $x = -1$.

Slope of the tangent line at $x = -1$ is $f'(-1)$.

$f'(x) = 1 \cdot (x^2 - 1) + (2 + x) 2x$

so $f'(-1) = ((-1)^2 - 1) + (2 - 1)(-2)$

$= 0 - 2$

$= -2$

When $x = -1$, $y = (2 + (-1))((-1)^2 - 1) = 0$

so $(-1, 0)$ is on the line.

Point slope equation:

$\boxed{y - 0 = -2(x+1)}$

or $\boxed{y = -2(x+1)}$

or $\boxed{y = -2x - 2}$

or $\boxed{y + 2x = -2}$

4. (5 points each) Find the derivatives of the following functions (don't simplify your answers):

a. $f(x) = 6x - \frac{6}{x} + \frac{5}{2\sqrt{x}}$ $= 6x - 6x^{-1} + \frac{5}{2}x^{-\frac{1}{2}}$

$$f'(x) = 6 - 6\cdot(-1)x^{-2} - \frac{5}{2}\cdot\frac{1}{2}x^{-\frac{3}{2}}$$

or $$6 + 6x^{-2} - \frac{5}{4}x^{-3/2}$$

b. $f(x) = (x^2 - x^3)(\frac{1}{x^2} + x)$ prod. rule

$$f'(x) = (2x - 3x^2)(\tfrac{1}{x^2} + x) + (x^2 - x^3)(x^{-2} + x)'$$

$$= (2x - 3x^2)(\tfrac{1}{x^2} + x) + (x^2 - x^3)(-2x^{-3} + 1)$$

c. $f(x) = \frac{x-1}{x+1}$ Quotient Rule

$$f'(x) = \frac{1\cdot(x+1) - (x-1)\cdot 1}{(x+1)^2} = \boxed{\frac{(x+1) - (x-1)}{(x+1)^2}} \quad \text{or} \quad \frac{2}{(x+1)^2}$$

Chain Rule

d. $f(x) = \sqrt[3]{x^3 + 2x}$ $= (x^3 + 2x)^{1/3}$

$$f'(x) = \frac{1}{3}(x^3 + 2x)^{-2/3}(3x^2 + 2)$$

5. (5 points each) Suppose that the revenue for selling x calculators is $R(x) = 100x + 0.05x^2$.

a. Find the rate of change of the revenue with respect to the number of calculators sold.

$R'(x) = 100 + 2(0.05)x$ or $R'(x) = 100 + 0.1x$

b. What is the rate of change of the revenue when 100 calculators are sold?

$R'(100) = 100 + 0.1 \times 100 = 110$

6. (5 points each) Evaluate the following:

$$\text{a. } \lim_{x\to\infty} \frac{2 - 3x^3}{2x^3 - x}$$

This will be the same as $\lim_{x\to\infty} \frac{-3x^3}{2x^3} = -\frac{3}{2}$

ignoring lower power terms in numerator + denominator.

$$\text{a. } \lim_{x\to-\infty} \frac{x^3 - 1}{x + 1}$$

This will be the same as

$\lim_{x\to-\infty} \frac{x^3}{x} = \lim_{x\to-\infty} \frac{x^2 \cdot x}{x} = \lim_{x\to-\infty} x^2 = \infty$