$\mathbb{Q}$–GORENSTEIN SPLINTER RINGS OF CHARACTERISTIC p ARE F–REGULAR

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1. Introduction

A Noetherian integral domain R is said to be a *splinter* if it is a direct summand, as an R–module, of every module–finite extension ring, see [Ma]. In the case that R contains the field of rational numbers, it is easily seen that R is splinter if and only if it is a normal ring, but the notion is more subtle for rings of characteristic $p > 0$. It is known that F–regular rings of characteristic p are splinters, and Hochster and Huneke showed that the converse is true for locally excellent Gorenstein rings, [HH4]. In this paper we extend their result by showing that $\mathbb{Q}$–Gorenstein splinters are F–regular. Our main theorem is:

Theorem 1.1. *Let R be a locally excellent $\mathbb{Q}$–Gorenstein integral domain of characteristic $p > 0$. Then R is F–regular if and only if it is a splinter.*

These issues are closely related to the question of whether the *tight closure* I^* of an ideal I of a characteristic p domain agrees with its *plus closure*, i.e., $I^+ = IR^+ \cap R$, where R^+ denotes the integral closure of R in an algebraic closure of its fraction field. We always have the containment $I^+ \subseteq I^*$, and Smith showed that equality holds if I is a parameter ideal in an excellent domain R, see [Sm1]. An excellent domain R of characteristic p is splinter if and only if for all ideals I of R, we have $I^+ = I$.

For an excellent local domain R of characteristic p, Hochster and Huneke showed that R^+ is a big Cohen–Macaulay algebra, see [HH2]. For further work on R^+ and plus closure see [Ab, AH]. Our main references for the theory of tight closure are [HH1, HH3, HH4].

Although tight closure is primarily a characteristic p notion, it has strong connections with the study of singularities of algebraic varieties over fields of characteristic zero. For $\mathbb{Q}$–Gorenstein rings essentially of finite type over a field of characteristic zero, it is known that F–regular type is equivalent to log–terminal singularities, see [Ha, Sm2, Sm3, Wa]. Consequently our main theorem offers a characterization of log–terminal singularities in characteristic zero, see Corollary 3.3.

2. Preliminaries

By the *canonical ideal* of a Cohen–Macaulay normal domain (R, m), we shall mean an ideal of R which is isomorphic to the canonical module of R. We next record some results that we shall use later in our work.

Lemma 2.1. *Let* (R, m) *be a Cohen–Macaulay local domain with canonical ideal* J. *Fix a system of parameters* $y_1, \dots, y_d$ *for* R, *and let* $s \in J$ *be an element which represents a socle generator in* $J/(y_1, \dots, y_d)J$. *Then for* $t \in \mathbb{N}$, *the element* $s(y_1 \dots y_d)^{t-1}$ *is a socle generator in* $J/(y_1^t, \dots, y_d^t)J$. *The ideals* $I_t = (y_1^t, \dots, y_d^t)J :_R s$ *form a family of irreducible ideals which are cofinal with the powers of the maximal ideal* m *of* R.

Proof. See the proof of [HH4, Theorem 4.6]. ☐

Lemma 2.2. *Let* R *be a Cohen–Macaulay normal domain with canonical ideal* J. *Pick* $y_1 \neq 0$ *in* J. *Then there exists an element* y_2 *not in any minimal prime of* y_1 *and* $\gamma \in J$ *such that* $y_2^i J^{(i)} \subseteq \gamma^i R$ *for all positive integers* i.

Proof. This is [Wi, Lemma 4.3]. ☐

Lemma 2.3. *Let* (R, m) *be a normal local domain and* J *an ideal of pure height one, which has order* n *when regarded as an element of the divisor class group* $\operatorname{Cl}(R)$. *Then for* $0 < i < n$, *we have* $J^{(i)}J^{(n-i)} \subseteq J^{(n)}m$.

Proof. Let $J^{(n)} = \alpha R$. Clearly $J^{(i)}J^{(n-i)} \subseteq \alpha R$, and it suffices to show that $J^{(i)}J^{(n-i)} \subsetneq \alpha R$. If $J^{(i)}J^{(n-i)} = \alpha R$, then $J^{(i)}$ is an invertible fractional ideal, and so must be a projective R–module. Since R is local, $J^{(i)}$ is a free R–module, but this is a contradiction since $J^{(i)}$ cannot be principal for $0 < i < n$. ☐

Discussion 2.4. Let (R, m) be a $\mathbb{Q}$–Gorenstein Cohen–Macaulay normal local domain, with canonical ideal J. Let n denote the order of J as an element of the divisor class group $\operatorname{Cl}(R)$, and pick $\alpha \in R$ such that $J^{(n)} = \alpha R$. Consider the subring $R[JT, J^{(2)}T^2, \dots]$ of $R[T]$, and let

$$S = R[JT, J^{(2)}T^2, \dots]/(\alpha T^n - 1).$$

Note that S has a natural $\mathbb{Z}/n\mathbb{Z}$–grading where $[S]_0 = R$, and for $0 < i < n$ we have $[S]_i = J^{(i)}T^i$. We claim that the ideal

$$\mathfrak{m} = m + JT + J^{(2)}T^2 + \cdots + J^{(n-1)}T^{n-1}$$

is a maximal ideal of S. Since each $J^{(i)}$ is an ideal of R, we need only verify that $J^{(i)}T^i\mathfrak{m} \subseteq \mathfrak{m}$ for $0 < i < n-1$, but this follows from Lemma 2.3. Note furthermore that $\mathfrak{m}^n \subseteq mS$.

3. The main result

Proof of Theorem 1.1. The property of being a splinter localizes, as does the property of being $\mathbb{Q}$–Gorenstein. Hence if the splinter ring R is not F–regular, we may localize at a prime ideal $P \in \operatorname{Spec} R$ which is minimal with respect to the property that R_P is not F–regular. After a change of notation, we have a splinter (R, m) which has an isolated non F–regular point at the maximal ideal m. This shows that R has an m–primary test ideal. However since R is a splinter it must be F–pure, and so the test ideal is precisely the maximal ideal m. Note that by [Sm1, Theorem 5.1] parameter ideals of R are tightly closed, and R is indeed F–rational.

Let $\dim R = d$. Choose a system of parameters for R as follows: first pick a nonzero element $y_1 \in J$. Then, by Lemma 2.2, pick y_2 not in any minimal prime of y_1 such that $y_2^i J^{(i)} \subseteq \gamma^i R$ for a fixed element $\gamma \in J$, for all positive integers i. Extend y_1, y_2 to a full system of parameters $y_1, \ldots, y_d$ for R. Since $y_1 \in J$, there exists $u \in R$ such that $s = uy_1$ is a socle generator in $J/(y_1, \ldots, y_d)J$. Let Y denote the product $y_1 \ldots y_d$.

Consider the family of ideals $\{I_c\}_{c \in \mathbb{N}}$ as in Lemma 2.1. If R is not F–regular, there exists an irreducible ideal $I_c = (y_1^c, \ldots, y_d^c)J :_R s$ which is not tightly closed, specifically $Y^{c-1} \in I_c^*$. Consequently $sY^{c-1} \in (y_1^c, \ldots, y_d^c)J^*$. In the ring S, this says that $sY^{c-1} \in (y_1^c, \ldots, y_d^c)JS^*$ and so

$$sTY^{c-1} \in (y_1^c, \ldots, y_d^c)JTS^* \subseteq (y_1^c, \ldots, y_d^c)S^*.$$

We shall first imitate the proof of [Sm1, Lemma 5.2] to obtain an "equational condition" from this statement. To simplify notation, let $z = sTY^{c-1}$ and $x_i = y_i^c$ for $1 \leq i \leq d$. We then have $z \in (x_1, \ldots, x_d)S^*$. Consider the maximal ideal $\mathfrak{m} = m + JT + J^{(2)}T^2 + \cdots + J^{(n-1)}T^{n-1}$ of S and the highest local cohomology module

$$H^d_{\mathfrak{m}}(S) = \varinjlim S/(x_1^i, \ldots, x_d^i),$$

where the maps in the direct limit system are induced by multiplication by $x_1 \cdots x_d$.

Since the test ideal of R is m, if Q_0 is a power of p greater than n, we have $\mathfrak{m}^{Q_0} z^q \in (x_1^q, \ldots, x_d^q)S$ for all $q = p^e$.

Let η denote $[z + (x_1, \ldots, x_d)S]$ viewed as an element of $H^d_{\mathfrak{m}}(S)$, and N be the S–submodule of $H^d_{\mathfrak{m}}(S)$ spanned by all $F^e(\eta)$ where $e \in \mathbb{N}$. Since $H^d_{\mathfrak{m}}(S)$ is an S–module with DCC, there exists e_0 such that the submodules generated by $F^{e_0}(N)$ and $F^{e'}(N)$ agree for all $e' \geq e_0$. Hence there exists an equation of the form

$$F^{e_0}(\eta) = a_1 F^{e_1}(\eta) + \cdots + a_k F^{e_k}(\eta)$$

with $a_1, \ldots, a_k \in S$ and $e_0 < e_1 \leq e_2 \leq \cdots \leq e_k$. If some a_i is not a unit, we may use suitably high Frobenius iterations on the equation above, and the fact that for $Q_0 \geq n$ we have $\mathfrak{m}^{Q_0} F^e(\eta) = 0$ for all $e \in \mathbb{N}$, to replace the above equation by one in which the coefficients which occur are indeed units. Hence we have an equation $F^e(\eta) = a_1 F^{e_1}(\eta) + \cdots + a_k F^{e_k}(\eta)$ where $e < e_1 \leq e_2 \leq \cdots \leq e_k$ and $a_1, \ldots, a_k$ are units. Let $q = p^e$, $q_i = p^{e_i}$ for $1 \leq i \leq k$ and $X = x_1 \ldots x_d$. Rewriting our equation we have

$$\begin{aligned}[z^q X^{q_k - q} + (x_1^{q_k}, \ldots, x_d^{q_k})S] \;&=\; a_1[z^{q_1} X^{q_k - q_1} + (x_1^{q_k}, \ldots, x_d^{q_k})S] \;+\; \cdots \\ &\qquad \cdots \;+\; a_k[z^{q_k} + (x_1^{q_k}, \ldots, x_d^{q_k})S],\end{aligned}$$

i.e., $[z^q X^{q_k - q} \;-\; a_1 z^{q_1} X^{q_k - q_1} \;-\; \cdots \;-\; a_k z^{q_k} \;+\; (x_1^{q_k}, \ldots, x_d^{q_k})S] = 0$. Since the ring S may not necessarily be Cohen–Macaulay, we cannot assume that the maps in the direct limit system $\varinjlim S/(x_1^i, \ldots, x_d^i)$ are injective. However for a suitable positive integer b we do obtain the equation

$$(zX^{b-1})^Q \;\in (x_1^{bQ}, \ldots,\; x_d^{bQ},\; zX^{bQ-1},\; z^p X^{bQ-p}, \ldots,\; z^{Q/p} X^{bQ-Q/p})S,$$

where $Q = q_k$. Going back to the earlier notation and setting $t = bc$, we have

$$(sTY^{t-1})^Q \;\in (y_1^{tQ}, \ldots, y_d^{tQ},\; sTY^{tQ-1},\; (sT)^p Y^{tQ-p}, \ldots, (sT)^{Q/p} Y^{tQ-Q/p})S.$$

Note that $\frac{1}{T} = \alpha T^{n-1} \in S$, and multiplying the above by $\frac{1}{T^Q}$, we get

$$\begin{aligned}(sY^{t-1})^Q \;\in\; (y_1^{tQ} \frac{1}{T^Q}, \ldots,\; y_d^{tQ} \frac{1}{T^Q},\; sY^{tQ-1} \frac{1}{T^{Q-1}},\; s^p Y^{tQ-p} \frac{1}{T^{Q-p}}, \ldots \\ \ldots, s^{Q/p} Y^{tQ-Q/p} \frac{1}{T^{Q-Q/p}})S.\end{aligned}$$

Since $(sY^{t-1})^Q \in [S]_0 = R$, we may intersect the ideal above with R to obtain

$$(sY^{t-1})^Q \in (y_1^{tQ}J^{(Q)}, \dots,\ y_d^{tQ}J^{(Q)},\ sY^{tQ-1}J^{(Q-1)},\ s^pY^{tQ-p}J^{(Q-p)}, \dots \\ \dots,\ s^{Q/p}Y^{tQ-Q/p}J^{(Q-Q/p)})R.$$

Replacing $s = uy_1$ above, we get

$$(uy_1Y^{t-1})^Q \ \in \ (y_1^{tQ}J^{(Q)},\ \dots,\ y_d^{tQ}J^{(Q)},\ (uy_1)Y^{tQ-1}J^{(Q-1)}, \\ (uy_1)^pY^{tQ-p}J^{(Q-p)},\ \dots,\ (uy_1)^{Q/p}Y^{tQ-Q/p}J^{(Q-Q/p)})R.$$

Let $Z = \frac{Y}{y_1} = y_2 \dots y_d$. We then have

$$(uZ^{t-1})^Q y_1^{tQ} \ \in \ (y_1^{tQ}J^{(Q)},\ y_2^{tQ},\ \dots,\ y_d^{tQ},\ uy_1^{tQ}Z^{tQ-1}J^{(Q-1)}, \\ u^py_1^{tQ}Z^{tQ-p}J^{(Q-p)},\ \dots,\ u^{Q/p}y_1^{tQ}Z^{tQ-Q/p}J^{(Q-Q/p)})R.$$

Using the fact that $y_1, \dots, y_d$ are a system of parameters for the Cohen–Macaulay ring R, we get

$$(uZ^{t-1})^Q \ \in \ (J^{(Q)},\ y_2^{tQ},\ \dots,\ y_d^{tQ},\ uZ^{tQ-1}J^{(Q-1)},\ u^pZ^{tQ-p}J^{(Q-p)}, \dots \\ \dots,\ u^{Q/p}Z^{tQ-Q/p}J^{(Q-Q/p)})R.$$

Consequently there exists $a \in J^{(Q)}$, $b_i \in R$ and $c_{p^e} \in J^{(Q-Q/p^e)}$ such that

$$(uZ^{t-1})^Q \ = \ a \ + \ \sum_{i=2}^{d} b_iy_i^{tQ} \ + \ c_1uZ^{tQ-1} \ + \ c_pu^pZ^{tQ-p} \ + \ \cdots \\ \cdots \ + \ c_{Q/p}u^{Q/p}Z^{tQ-Q/p}.$$

For $2 \le i \le d$, consider the following equations in the variables $V_2, \dots, V_d$:

$$V_i^Q = b_i + c_1V_i(\frac{Z}{y_i})^{tQ-t} + c_pV_i^p(\frac{Z}{y_i})^{tQ-tp} + \cdots + c_{Q/p}V_i^{Q/p}(\frac{Z}{y_i})^{tQ-tQ/p}.$$

Since these are monic equations defined over R, there exists a module finite normal extension ring R_1, with solutions v_i of these equations. Working in the ring R_1, let

$$w = uZ^{t-1} - \sum_{i=2}^{d} v_iy_i^t.$$

Combining the earlier equations, we have

$$w^Q = a + c_1wZ^{tQ-t} + c_pw^pZ^{tQ-tp} + \cdots + c_{Q/p}w^{Q/p}Z^{tQ-tQ/p}.$$

Multiplying this equation by y_2^Q and using the fact that $y_2^iJ^{(i)} \subseteq \gamma^iR$ for all positive integers i, we get

$$(wy_2)^Q = d_0\gamma^Q + d_1wy_2\gamma^{Q-1} + d_p(wy_2)^p\gamma^{Q-p} + \cdots + d_{Q/p}(wy_2)^{Q/p}\gamma^{Q-Q/p}.$$

Since the ring R_1 is normal, The above equation gives an equation by which wy_2/γ is integral over the ring R_1. Since R_1 is normal, we have $wy_2 \in \gamma R_1$. Combining this with $w = uZ^{t-1} - \sum_{i=2}^{d} v_i y_i^t$, we have

$$uZ^{t-1}y_2 = wy_2 + (\sum_{i=2}^{d} v_i y_i^t)y_2 \ \in \ (J,\ y_2^{t+1},\ y_2y_3^t,\ \dots,\ y_2y_d^t)R_1,$$

and so

$$uZ^{t-1}y_2 \ \in \ (J,\ y_2^{t+1},\ y_2y_3^t, \dots,\ y_2y_d^t)^+ = (J,\ y_2^{t+1},\ y_2y_3^t, \dots,\ y_2y_d^t)R.$$

Since y_2 is not in any minimal prime of J, we get $uZ^{t-1} \in (J, y_2^t, y_3^t, \dots, y_d^t)R$. Multiplying this by y_1, we get

$$sZ^{t-1} \ \in \ (y_1J,\ y_1y_2^t,\ y_1y_3^t,\ \dots,\ y_1y_d^t)R \subseteq (y_1,\ y_2^t,\ y_3^t,\ \dots,\ y_d^t)J,$$

but this contradicts the fact that s generates the socle in $J/(y_1, \dots, y_d)J$. □

Corollary 3.1. *Let* (R, m) *be an excellent integral domain of dimension two over a field of characteristic* $p > 0$*. Then* R *is a splinter if and only if it is F–regular.*

Proof. The hypotheses imply that R is F–rational, and so has a torsion divisor class group by a result of Lipman, [Li]. Hence R must be $\mathbb{Q}$–Gorenstein. □

Definition 3.2. Let $R = K[X_1, \dots, X_n]/I$ be a domain finitely generated over a field K of characteristic zero. We say R is of *splinter type* if there exists a finitely generated $\mathbb{Z}$–algebra $A \subseteq K$ and a finitely generated free A–algebra $R_A = A[X_1, \dots, X_n]/I_A$ such that $R \cong R_A \otimes_A K$, and for all maximal ideals μ in a Zariski dense subset of Spec A, the fiber rings $R_A \otimes_A A/\mu$ (which are rings over fields of characteristic p) are splinter.

Using the equivalence of F–regular type and log–terminal singularities for rings finitely generated over a field of characteristic zero (see [Ha, Sm3, Wa]) we obtain the following corollary:

Corollary 3.3. *Let* R *be a finitely generated* $\mathbb{Q}$*–Gorenstein domain over a field of characteristic zero. Then* R *has log–terminal singularities if and only if it is of splinter type.*

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