

Multi-symbolic Rees algebras and strong F-regularity

Anurag K. Singh

Department of Mathematics, University of Utah, 155 S. 1400 E., Salt Lake City, UT 84112-0090, USA
e-mail: singh@math.utah.edu

Received: August 10, 1998

Abstract Let I be a divisorial ideal of a strongly F-regular ring A . K.-i. Watanabe raised the question whether the symbolic Rees algebra $\mathcal{R}_s(I) = \bigoplus_{n \geq 0} I^{(n)}$ is Cohen-Macaulay whenever it is Noetherian. We develop the notion of multi-symbolic Rees algebras and use this to show that $\mathcal{R}_s(I)$ is indeed Cohen-Macaulay whenever a certain auxiliary ring is finitely generated over A .

1 Introduction

In [Wa] K.-i. Watanabe raised the issue whether for a divisorial ideal I of a strongly F-regular ring A , the symbolic Rees algebra $\mathcal{R}_s(I) = \bigoplus_{n \geq 0} I^{(n)}U^n$ is Cohen-Macaulay whenever it is Noetherian. Watanabe showed that this is true when I is an *anti-canonical ideal* i.e., an ideal of pure height one which represents the inverse of the class of the canonical module of A in the divisor class group $\text{Cl}(A)$. In this paper we work in the more general setting of multi-symbolic Rees algebras and as a corollary of our main result, Theorem 5.1, we obtain the following positive answer to Watanabe's question:

Theorem 1.1 *Let (A, m) be a strongly F-regular ring with canonical ideal ω . Given an ideal I of A of pure height one, choose J of pure height one such that $[I] + [J] + [\omega] = 0$ in $\text{Cl}(A)$. If the multi-symbolic Rees algebra $\mathcal{R}_s(I, J)$ is finitely generated over A , then $\mathcal{R}_s(I)$ is Cohen-Macaulay.*

The hypothesis that A is strongly F-regular is indeed used in an essential way: Watanabe has constructed an example of an F-rational

ring A with a divisorial ideal I such that the symbolic Rees algebra $\mathcal{R}_s(I)$ is not Cohen-Macaulay, see [Wa, Example 4.4].

In general, of course, the symbolic Rees algebra $\mathcal{R}_s(I)$ for a divisorial ideal I of a normal ring A need not be Noetherian, e.g., if A is the coordinate ring of an elliptic curve, and I is a prime ideal of height one, which has infinite order in the divisor class group, $\text{Cl}(A)$. However if we specialize to the case when A is F -rational, a two-dimensional example is easily ruled out since, by a result of J. Lipman, [Li], the divisor class group of a two dimensional rational singularity (and hence by [Sm2] of a two dimensional F -rational ring) is a torsion group. In dimension three the hypothesis that A has rational singularities is no longer sufficient: S. D. Cutkosky has shown that a symbolic Rees algebra over a three dimensional ring with rational singularities need not be Noetherian, see [Cu, Theorem 6]. It should be noted that if A is a Gorenstein ring of dimension three over \mathbb{C} with rational singularities, then symbolic Rees algebras at divisorial ideals of A are finitely generated by [Ka, Theorem 6.1].

2 Preliminaries

Throughout our discussion all rings are commutative and have a unit element. Unless stated otherwise, we shall assume our rings contain a field K of characteristic $p > 0$. We use the letter e to denote a variable nonnegative integer, and q to denote the e th power of p . We denote by F the Frobenius endomorphism of A , i.e., $F(a) = a^p$. For a reduced ring A of characteristic $p > 0$, $A^{1/q}$ shall denote the ring obtained by adjoining all q th roots of elements of A . The ring A is said to be F -finite if $A^{1/p}$ is module-finite over A . Note that a finitely generated algebra A over a field K is F -finite if and only if $K^{1/p}$ is a finite field extension of K .

In the notation (A, m) , the ring A is either a Noetherian local ring with maximal ideal m , or an \mathbb{N} -graded ring with homogeneous maximal ideal $m = \bigoplus_{i>0} A_i$ which is finitely generated over a field $A_0 = K$.

By a normal domain, we shall mean a Noetherian domain which is integrally closed in its field of fractions.

Our references for the theory of tight closure are [HH1], [HH2], [HH3], and [HH4]. We next recall some definitions and well known facts.

Definition 2.1 *A ring A is said to be F -pure if for all A -modules M , the Frobenius homomorphism $F : M \rightarrow F(M)$ is injective.*

An F-finite domain A is strongly F-regular if for every nonzero element $c \in A$, there exists $q = p^e$ such that the A -linear inclusion $A \rightarrow A^{1/q}$ sending 1 to $c^{1/q}$ splits as a map of A -modules.

Regular rings are strongly F-regular, and strongly F-regular rings are Cohen-Macaulay. If B is a strongly F-regular ring and A is a subring which is a direct summand of B as an A -module, then A is also strongly F-regular.

Let (A, m) be an F-finite local domain, and let $E = E_A(A/m)$ denote the injective hull of the residue field A/m . K. E. Smith has shown that A is strongly F-regular if and only if the zero submodule of E is tightly closed, see [Sml, Proposition 7.1.2]. Hence if $\zeta \in E$ is a socle generator, A is strongly F-regular if and only if for every nonzero element $c \in R$, there exists a positive integer e such that $cF^e(\zeta) \neq 0$.

Definition 2.2 Let A be a normal domain and I an ideal of pure height one. Then $I^{(n)}$ denotes the n th symbolic power of the ideal I , i.e., the reflexive hull of I^n . If \mathcal{F} is the set of minimal primes of I , we have

$$I^{(n)} = \left(\bigcap_{P \in \mathcal{F}} I^n A_P \right) \cap A.$$

3 Multi-symbolic Rees algebras

Let (A, m) be a Noetherian normal local domain, and I an ideal of pure height one. The symbolic Rees algebra

$$\mathcal{R}_s(I) = \bigoplus_{n \geq 0} I^{(n)} U^n \subseteq A[U]$$

is an object that has been studied extensively. We generalize this construction to a finite family of ideals I_1, I_2, \dots, I_k , each of pure height one, by defining

$$\mathcal{R}_s(I_1, I_2, \dots, I_k) = \bigoplus_{n_1, \dots, n_k \in \mathbb{N}} (I_1^{n_1} I_2^{n_2} \dots I_k^{n_k})^{**} U_1^{n_1} U_2^{n_2} \dots U_k^{n_k}$$

as a subring of the polynomial ring $A[U_1, \dots, U_k]$. Here $*$ denotes the dual $\text{Hom}_A(-, A)$. In this notation $(I_1^{n_1} \dots I_k^{n_k})^{**}$ is the reflexive hull of $I_1^{n_1} \dots I_k^{n_k}$.

Proposition 3.1 Let (A, m) be a normal domain and I_1, \dots, I_k be ideals of A of pure height one. Then the multi-symbolic Rees algebra $B = \mathcal{R}_s(I_1, \dots, I_k)$ is a Krull domain. Hence the ring B is a normal domain whenever it is Noetherian.

Proof Let \mathcal{F} be the set of all minimal prime ideals of I_1, \dots, I_k . We then have

$$\mathcal{R}_s(I_1, \dots, I_k) = \left(\bigcap_{P \in \mathcal{F}} A[I_1 A_P U_1, \dots, I_k A_P U_k] \right) \cap A[U_1, \dots, U_k]$$

which, being a finite intersection of Krull domains, is a Krull domain.

□

Proposition 3.2 *Let (A, m) be a normal domain, and I_1, \dots, I_k be ideals of A of pure height one such that the multi-symbolic Rees algebra $B = \mathcal{R}_s(I_1, I_2, \dots, I_{k-1})$ is Noetherian. Let $\tilde{I}_k = (I_k B)^{**}$ denote the reflexivization of $I_k B$ as a B -module, i.e., $*$ denotes $\text{Hom}_B(-, B)$. Then there is a natural isomorphism*

$$\mathcal{R}_s(\tilde{I}_k) = B \oplus \tilde{I}_k \oplus \tilde{I}_k^{(2)} \oplus \dots \cong \mathcal{R}_s(I_1, I_2, \dots, I_k).$$

Proof There is a natural inclusion

$$B \oplus I_k U_k B \oplus I_k^2 U_k^2 B \oplus \dots \rightarrow \mathcal{R}_s(I_1, \dots, I_k).$$

To obtain the isomorphism asserted, we need to verify that a reflexive B -module is reflexive when considered as an A -module. For this it suffices to verify that B is a reflexive A -module, but this follows since B is a direct sum of reflexive A -modules. □

This gives us the immediate corollary:

Corollary 3.3 *Let (A, m) be a normal domain, and I_1, \dots, I_k ideals of A of pure height one such that $B = \mathcal{R}_s(I_1, I_2, \dots, I_k)$ is Noetherian. Then B arises by a successive construction of symbolic Rees algebras starting with the ring A .*

Theorem 3.4 *Let (A, m) be a normal domain, and I_1, \dots, I_k be ideals of A of pure height one such that the multi-symbolic Rees algebra $B = \mathcal{R}_s(I_1, I_2, \dots, I_k)$ is Noetherian. Then the inclusion $A \subseteq B$ satisfies Samuel's PDE condition, i.e., for a height one prime $P \in \text{Spec } B$, we have $\text{height}(P \cap A) \leq 1$. This gives a natural map of divisor class groups, $i : \text{Cl}(A) \rightarrow \text{Cl}(B)$, which is an isomorphism.*

Furthermore if A and B are homomorphic images of regular local rings and ω_A and ω_B denote the canonical modules of A and B respectively, we have the relation

$$[\omega_B] = i([\omega_A] + [I_1] + \dots + [I_k]).$$

Proof The corresponding statements for symbolic Rees algebras (i.e., the case $k = 1$) are covered by [GHN, Lemma 4.3 (2), Proposition 4.4, Theorem 4.5], see also [ST, Proposition 2.6]. The assertions here follow by combining these results with Corollary 3.3 above. \square

Determining when symbolic Rees algebras are Cohen-Macaulay is a subtle issue: as remarked earlier, Watanabe has constructed examples where A is a ring with rational singularities and I is an anti-canonical ideal, but the symbolic Rees ring $\mathcal{R}_s(I)$ is not Cohen-Macaulay. However if the divisorial ideals I_1, \dots, I_k have finite order as elements of the divisor class group $\text{Cl}(A)$, we have the following extension of [GHN, Theorem 4.1]:

Theorem 3.5 *Let (A, m) be a normal domain, and I_1, \dots, I_k be ideals of pure height one which have finite order as elements of the divisor class group $\text{Cl}(A)$. Then the multi-symbolic Rees algebra $B = \mathcal{R}_s(I_1, \dots, I_k)$ is Cohen-Macaulay if and only if for all $n_i \in \mathbb{N}$ the ideals $(I_1^{n_1} \cdots I_k^{n_k})^{**}$ are maximal Cohen-Macaulay A -modules.*

Proof Let a_i denote the order of $[I_i]$ in $\text{Cl}(A)$, and fix elements x_i such that $I_i^{(a_i)} = x_i A$, for $1 \leq i \leq k$. The elements $x_1 U^{a_1}, \dots, x_k U^{a_k}$ form part of a system of parameters for B , and it is easily verified that this is a regular sequence on B . Next note that

$$B/(x_1 U^{a_1}, \dots, x_k U^{a_k}) = \bigoplus_{0 \leq n_{ij} < a_i} (I_1^{n_{1j}} \cdots I_k^{n_{kj}})^{**} U_1^{n_{1j}} \cdots U_k^{n_{kj}}$$

and so B is a Cohen-Macaulay ring if and only every system of parameters for A is a regular sequence on the ideals $(I_1^{n_1} I_2^{n_2} \cdots I_k^{n_k})^{**}$ for all $n_1, \dots, n_k \in \mathbb{N}$. \square

4 Examples

Example 4.1 Consider the subring $A = K[ax, ay, bx, by]$ of the polynomial ring $K[a, b, x, y]$ and the height one prime ideals $P_i = (ax, ay)$ and $Q_j = (ax, bx)$ where $1 \leq i \leq n$ and $1 \leq j \leq m$. Then the multi-symbolic Rees algebra

$$B = A(P_1, \dots, P_n, Q_1, \dots, Q_m)$$

is isomorphic to the Segre product of two polynomial rings,

$$K[X_1, \dots, X_{n+2}] \# K[Y_1, \dots, Y_{m+2}].$$

We have $\text{Cl}(A) = \text{Cl}(B) = \mathbb{Z}$, and

$$[\omega_B] = i(n[P] + m[Q]) = i((n - m)[P]).$$

In particular, B is Gorenstein if and only if $n = m$.

Example 4.2 Let $A = K[X_1, \dots, X_m]^{(n)}$ denote the n th Veronese subring of the polynomial ring $K[X_1, \dots, X_m]$. We compute all multi-symbolic Rees algebras over the ring A . The divisor class group of A is $\text{Cl}(A) = \mathbb{Z}/n\mathbb{Z}$ and we fix as a generator, the height one prime ideal

$$P = (X_1 K[X_1, \dots, X_m]) \cap A$$

Divisorial ideals of A are of the form $P^{(i)}$ up to isomorphism, and the multi-symbolic Rees algebra $A(P^{(\alpha_1)}, \dots, P^{(\alpha_k)})$ is determined by the k -tuple of integers $\alpha_1, \dots, \alpha_k$. We claim that $A(P^{(\alpha_1)}, \dots, P^{(\alpha_k)})$ is isomorphic to the n th Veronese subring of the polynomial ring

$$K[X_1, \dots, X_m, X_1^{\alpha_1} U_1, \dots, X_1^{\alpha_k} U_k]$$

where the variables U_1, \dots, U_k have weight zero. To see this, note that by definition we have

$$A(P^{(\alpha_1)}, \dots, P^{(\alpha_k)}) = \bigoplus_{n_i \geq 0} P^{(n_1 \alpha_1 + \dots + n_k \alpha_k)} U_1^{n_1} \dots U_k^{n_k}$$

and that a monomial in X_1, \dots, X_m is an element of $P^{(n_1 \alpha_1 + \dots + n_k \alpha_k)}$ precisely if it is a multiple of $X_1^{n_1 \alpha_1 + \dots + n_k \alpha_k}$ whose degree is a multiple of n .

5 An application to tight closure theory

Let (A, m) be a strongly F -regular domain with canonical module ω . In [Wa] Watanabe showed that the anti-canonical symbolic Rees algebra $\mathcal{R}_s(I)$ is Cohen-Macaulay (in fact, strongly F -regular) whenever it is Noetherian, and raised the question whether this is true for an arbitrary ideal I of pure height one. As an application of the construction of multi-symbolic Rees algebras, we show that $\mathcal{R}_s(I)$ is strongly F -regular, and in particular is Cohen-Macaulay, whenever a certain auxiliary algebra is finitely generated over A . Our main theorem is:

Theorem 5.1 *Let (A, m) be an F -finite normal ring with canonical ideal ω . Given an ideal I of A of pure height one, choose J of pure height one such that $[I] + [J] + [\omega] = 0$ in the divisor class group $\text{Cl}(A)$. Assume that the multi-symbolic Rees algebra $\mathcal{R}_s(I, J)$ is finitely generated over A . If A is F -pure, then the rings $\mathcal{R}_s(I)$ and $\mathcal{R}_s(I, J)$ are also F -pure. If A is strongly F -regular, then $\mathcal{R}_s(I)$ and $\mathcal{R}_s(I, J)$ are strongly F -regular, and in particular, are Cohen-Macaulay.*

Proof Let $B = \mathcal{R}_s(I) = \bigoplus_{i \geq 0} I^{(i)}U^i$ and $\tilde{J} = (JB)^{**}$ where $*$ denotes $\text{Hom}_B(-, B)$. If $R = \mathcal{R}_s(I, J) \subseteq A[U, V]$, by Proposition 3.2 we have

$$R = \mathcal{R}_s(\tilde{J}) = B \oplus \tilde{J} \oplus \tilde{J}^{(2)} \oplus \dots$$

Setting $d = \dim A$, we have $\dim R = d + 2$. Consider the maximal ideal of B ,

$$\mathfrak{m} = m + IU + I^{(2)}U^2 + \dots$$

and the maximal ideal of R ,

$$\mathfrak{M} = \mathfrak{m} + \tilde{J}V + \tilde{J}^{(2)}V^2 + \dots$$

In [Wa, Theorem 2.2] Watanabe has computed the highest local cohomology module of a symbolic Rees ring, and furthermore determined the Frobenius action on it. Using this we have

$$H_{\mathfrak{M}}^{d+2}(R) \cong \bigoplus_{j < 0} H_{\mathfrak{m}}^{d+1}(\tilde{J}^{(j)})V^j.$$

Again using Watanabe's result we get

$$H_{\mathfrak{M}}^{d+2}(R) \cong \bigoplus_{i < 0, j < 0} H_m^d(I^{(i)}J^{(j)})U^iV^j.$$

By Theorem 3.4 and the fact that $[I] + [J] + [\omega] = 0$ in $\text{Cl}(A)$ we see that $R = A(I, J)$ is quasi-Gorenstein, i.e., has a trivial canonical module. Hence $H_{\mathfrak{M}}^{d+2}(R)$ is the injective hull of R/\mathfrak{M} , and so the strong F-regularity or F-purity of R can be determined by studying the action of the Frobenius on $H_{\mathfrak{M}}^{d+2}(R)$.

Let ζ be a socle generator of $H_m^d(\omega)$. Then the socle of $H_{\mathfrak{M}}^{d+2}(R)$ is generated by $\zeta U^{-1}V^{-1}$. If A is F-pure, then $F(\zeta) \neq 0$ and so

$$F(\zeta U^{-1}V^{-1}) = F(\zeta)U^{-p}V^{-p} \neq 0,$$

by which R is F-pure. Consequently $B = \mathcal{R}_s(I)$, being a direct summand of R , is also F-pure.

Next assume that A is strongly F-regular. Let $cU^nV^m \in R$ be a nonzero element where $c \in (I^n J^m)^{**}$. To show that R is also strongly F-regular, it suffices to show that $(cU^nV^m) F^e(\zeta U^{-1}V^{-1}) \neq 0$ for some positive integer e . Choosing a suitable multiple, if necessary, we may assume that $n = m$. We may choose a canonical module ω for A such that $(IJ)^{**} = \omega^{(-1)}$. Then $c \in (I^n J^n)^{**} = \omega^{(-n)}$. Since A is strongly F-regular we may choose e such that $q = p^e > n$ and $cF^e(\zeta) \neq 0$ in $H_m^d(\omega^{(q)})$. But then $cF^e(\zeta) \neq 0$ as an element of $H_m^d(\omega^{(q-n)})$. Hence

$$cU^nV^n \cdot F^e(\zeta U^{-1}V^{-1}) = cF^e(\zeta)(UV)^{n-q} \in H_m^d(\omega^{(q-n)})(UV)^{n-q}$$

is nonzero, and so R is strongly F -regular. Hence its direct summand B is also strongly F -regular, and therefore is Cohen-Macaulay. \square

6 Rees rings

So far in our discussion, we had been considering symbolic Rees rings at ideals of pure height one. In this section we switch to the other extreme and consider the Rees ring at the homogeneous maximal ideal m of an \mathbb{N} -graded normal ring (A, m) . (In this case the Rees ring $R = A[mT]$ agrees with the symbolic Rees ring $\mathcal{R}_s(m)$.) Although the relation between the properties of A and those of $R = A[mT]$ seems to be very mysterious, there is one case where easy answers are available:

Proposition 6.1 *Let (A, m) be an \mathbb{N} -graded normal ring which is generated by its degree one elements over the field $A_0 = K$. Consider the Rees ring $R = A[mT]$. If A is strongly F -regular, F -pure, or a ring of characteristic zero with rational singularities, then the same is true for R .*

Proof Note that $R = A[mT]$ is isomorphic to the Segre product $A\#B$ where $B = K[S, T]$ is a polynomial ring in two variables. Consequently R is a direct summand of $A[S, T]$. If A has rational singularities, then so does $A[S, T]$, and consequently R has rational singularities (in characteristic 0) by Boutot's result, [Bo]. Similarly if A is strongly F -regular or F -pure, the same is true for $A[S, T]$, and its direct summand, R . \square

In the following example A is a normal monomial ring, i.e., a normal subring of a polynomial ring which is generated by monomials. Consequently A is strongly F -regular but we shall see that the Rees ring $A[mT]$ fails to be normal.

Example 6.2 Consider the monomial ring

$$A = K[W^3X, X^3Y, Y^3Z, Z^3W, W^2X^2Y^2Z^2] \subseteq K[W, X, Y, Z]$$

where K is a field. It is not difficult to see that A is isomorphic to the hypersurface

$$K[U_0, U_1, U_2, U_3, U_4]/(U_0^2 - U_1U_2U_3U_4)$$

and is a normal ring. Consequently by the main result of [Ho], A is a direct summand of a regular ring, and so is strongly F -regular. Take

the Rees ring be $R = A[mT]$. The element $W^2X^2Y^2Z^2T^2$ is in the fraction field of R , although it is not in R itself. However

$$(W^2X^2Y^2Z^2T^2)^2 = (W^3XT)(X^3YT)(Y^3ZT)(Z^3WT) \in R$$

and so R is not normal. Furthermore, when the characteristic of the field K is 2, it is easily verified that R is not F-pure, although the ring A is F-pure.

In the next example A is an F-rational hypersurface, but the Rees ring $A[mT]$, while being Gorenstein and normal, is not F-rational.

Example 6.3 Let $A = K[W, X, Y, Z]/(W^2 + X^3 + Y^6 + Z^7)$. Then A is F-rational whenever the characteristic of K is $p \geq 7$. We show that the Rees ring $R = A[mT]$, while being Gorenstein and normal, is not F-rational.

First note that the Rees ring R is Gorenstein. This holds, for example, by [GS, Theorem 1.2] since the associated graded ring

$$\mathrm{gr}_m(R) \cong K[W, X, Y, Z]/(W^2)$$

is Gorenstein with a -invariant $a(\mathrm{gr}_m(R)) = -2$.

We next examine R on the punctured spectrum. For $f \in m$, the localization $R_f \cong A_f[T]$ is a polynomial ring over A_f . For an element fT with $f \in m$, note that

$$R_{fT} \cong K\left[\frac{w}{f}, \frac{x}{f}, \frac{y}{f}, \frac{z}{f}, f, fT, \frac{1}{fT}\right].$$

Examining these localizations as f ranges through the set $\{w, x, y, z\}$, we can see that R is indeed normal.

To see that R is not F-rational, take $f = z$ above and let $U_1 = \frac{w}{z}$, $U_2 = \frac{x}{z}$ and $U_3 = \frac{y}{z}$. Then $R_{zT} \cong S[zT, 1/zT]$ where

$$S = K[U_1, U_2, U_3, Z]/(U_1^2 + U_2^3Z + U_3^6Z^4 + Z^5).$$

It suffices to show that S is not F-rational. Consider the grading on S where the variables U_1, U_2, U_3, Z have weights 15, 8, 1, 6 respectively. The a -invariant of S is easily computed to be $a(S) = 0$, and so S cannot be F-rational.

Acknowledgements It is a pleasure to thank Mel Hochster for several valuable discussions.

References

- [Bo] Boutot, J.-F.: Singularités rationnelles et quotients par les groupes réductifs. *Invent. Math.* **88**, 65–68 (1987)
- [Cu] Cutkosky, S. D.: Weil divisors and symbolic algebras. *Duke Math. J.* **57**, 175–183 (1988)
- [GHNV] Goto, S., Hermann, M., Nishida, K., Villamayor, O.: On the structure of Noetherian symbolic Rees algebras. *Manuscripta Math.* **67**, 197–225 (1990)
- [GS] Goto, S., Shimoda, Y.: On the Rees algebras of Cohen-Macaulay local rings. In: *Commutative algebra (Lecture Notes in Pure and Appl. Math., vol. 68, pp. 201–231)* New York: Dekker 1982
- [Ho] Hochster, M.: Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes. *Ann. of Math. (2)* **96**, 318–337 (1972)
- [HH1] Hochster, M., Huneke, C.: Tight closure and strong F-regularity. *Mem. Soc. Math. France* **38**, 119–133 (1989)
- [HH2] Hochster, M., Huneke, C.: Tight closure, invariant theory, and the Briançon-Skoda theorem. *J. Amer. Math. Soc.* **3**, 31–116 (1990)
- [HH3] Hochster, M., Huneke, C.: F-regularity, test elements, and smooth base change. *Trans. Amer. Math. Soc.* **346**, 1–62 (1994)
- [HH4] Hochster, M., Huneke, C.: Tight closure of parameter ideals and splitting in module-finite extensions. *J. Algebraic Geom.* **3**, 599–670 (1994)
- [Ka] Kawamata, Y.: Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces. *Ann. of Math. (2)* **127**, 93–163 (1988)
- [Li] Lipman, J.: Rational singularities, with applications to algebraic surfaces and unique factorization. *Inst. Hautes Études Sci. Publ. Math.* **36**, 195–279 (1969)
- [ST] Simis, A., Trung, N.V.: The divisor class group of ordinary and symbolic blow-ups. *Math. Z.* **198**, 479–491 (1988)
- [Sml] Smith, K.E.: Tight closure of parameter ideals and F-rationality. Thesis, Univ. of Michigan, 1993.
- [Sm2] Smith, K.E.: F-rational rings have rational singularities. *Amer. J. Math.* **119**, 159–180 (1997)
- [Wa] Watanabe, K.-i.: Infinite cyclic covers of strongly F-regular rings. *Contemp. Math.* **159**, 423–432 (1994)