A CONNECTEDNESS RESULT IN POSITIVE CHARACTERISTIC

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Dedicated to Professor Paul Roberts on the occasion of his sixtieth birthday

Abstract. Let \((R, \mathfrak{m})\) be a complete local ring of dimension at least two, which contains a separably closed coefficient field of positive characteristic. Using a vanishing theorem of Peskine-Szpiro, Lyubeznik proved that the local cohomology module \(H^1_{\mathfrak{m}}(R)\) is Frobenius-torsion if and only if the punctured spectrum of \(R\) is connected in the Zariski topology. We give a simple proof of this theorem and, more generally, a formula for the number of connected components in terms of the Frobenius action on \(H^1_{\mathfrak{m}}(R)\).

1. Introduction

All rings considered in this note are commutative and Noetherian. We give a simple proof of the following result due to Lyubeznik:

**Theorem 1.1** ([LY2, Corollary 4.6]). Let \((R, \mathfrak{m})\) be a complete local ring of dimension at least two, with a separably closed coefficient field of positive characteristic. Then the \(e\)-th iteration of the Frobenius map

\[ F: H^1_{\mathfrak{m}}(R) \to H^1_{\mathfrak{m}}(R) \]

is zero for \(e \gg 0\) if and only if \(\text{Spec } R \setminus \{\mathfrak{m}\}\) is connected in the Zariski topology.

We also obtain, by similar methods, the following theorem:

**Theorem 1.2.** Let \((R, \mathfrak{m})\) be a complete local ring of positive dimension, with an algebraically closed coefficient field of positive characteristic. Then the number of connected components of \(\text{Spec } R \setminus \{\mathfrak{m}\}\) is

\[ 1 + \dim_K \bigcap_{e \in \mathbb{N}} F^e(H^1_{\mathfrak{m}}(R)). \]

In Section 5 we describe how this provides an algorithm to determine the number of geometrically connected components of projective algebraic sets defined over a finite field: computer algebra algorithms for primary decomposition can be used to determine the number of connected components over finite extensions of the fields \(\mathbb{F}_p\) or \(\mathbb{Q}\), but not over the algebraic closures of these fields. In the case of characteristic zero, de Rham cohomology allows for the computation of the number...
of geometrically connected components via $D$-module methods, \cite{Wal}, and we show that the Frobenius provides analogous methods in the case of positive characteristic.

Theorem 1.1 is obtained in \cite{Ly2} as a corollary of the following two theorems of Lyubeznik and Peskine-Szpiro:

\textbf{Theorem 1.3 (\cite{Ly2} Theorem 1.1)}. Let $(A, \mathfrak{M})$ be a regular local ring containing a field of positive characteristic, and let $\mathfrak{A}$ be an ideal of $A$. Then $H^i_{\mathfrak{M}}(A) = 0$ if and only if there exists an integer $e \geq 1$ such that the $e$-th Frobenius iteration $F^e : H^{\dim A - i}(A/\mathfrak{A}) \to H^{\dim A - i}(A/\mathfrak{A})$ is the zero map.

\textbf{Theorem 1.4 (\cite{PS}, Chapter III, Theorem 5.5)}. Let $(A, \mathfrak{M})$ be a complete regular local ring with a separably closed coefficient field of positive characteristic, and let $\mathfrak{A}$ be an ideal of $A$. Then $H^i_{\mathfrak{M}}(A) = 0$ for $i \geq \dim A - 1$ if and only if $\dim(A/\mathfrak{A}) \geq 2$ and $\text{Spec}(A/\mathfrak{A}) \setminus \{\mathfrak{M}\}$ is connected.

Our proof of Theorem 1.1 is “simple” in the sense that it does not rely on vanishing theorems such as those of \cite{PS}—indeed, the only ingredient, aside from elementary considerations, is the local duality theorem. Results analogous to Theorem 1.4 were proved by Hartshorne in the projective case \cite[Theorem 7.5]{HaR}, and by Ogus in equicharacteristic zero using de Rham cohomology \cite[Corollary 2.1]{Og}. Combining these results, one has:

\textbf{Theorem 1.5}. Let $(A, \mathfrak{M})$ be a regular local ring containing a field, and let $\mathfrak{A}$ be an ideal of $A$. Then $H^i_{\mathfrak{M}}(A) = 0$ for $i \geq \dim A - 1$ if and only if

1. $\dim(A/\mathfrak{A}) \geq 2$, and
2. $\text{Spec}(A/\mathfrak{A}) \setminus \{\mathfrak{M}\}$ is formally geometrically connected (see Definition 2.1).

Huneke and Lyubeznik \cite[Theorem 2.9]{HL} gave a characteristic free proof of this using a generalization of a result of Faltings, \cite[Satz 1]{Fa}. Some other applications of local cohomology theory which yield strong results on the connectedness properties of algebraic varieties may be found in the papers \cite{BR} and \cite{HH}, where the authors obtain generalizations of Faltings’ connectedness theorem.

For the convenience of the reader, we include an Appendix with some facts about Frobenius actions; see Section 6.

\section*{2. Preliminary remarks}

\textit{Notation.} When $R$ is the homomorphic image of a ring $A$, we use upper-case letters $\mathfrak{P}, \Omega, \mathfrak{M}, \mathfrak{A}, \mathfrak{B}$ for ideals of $A$, and corresponding lower-case letters $p, q, m, a, b$ for their images in $R$.

\textbf{Definition 2.1}. Let $(R, m)$ be a local ring. A field $K \subseteq R$ is a coefficient field for $R$ if the composition $K \hookrightarrow R \twoheadrightarrow R/m$ is an isomorphism. Every complete local ring containing a field has a coefficient field.

We recall some notions from \cite[Chapitre VIII]{Ra}. Let $(R, m, K)$ be a local ring and let $f(T) \in K[T]$ denote the image of a polynomial $f(T) \in R[T]$. Then $R$ is Henselian if for every monic polynomial $f(T) \in R[T]$, every factorization of $f(T)$ as a product of relatively prime monic polynomials in $K[T]$ lifts to a factorization of $f(T)$ as a product of monic polynomials in $R[T]$. Hensel’s Lemma is precisely the statement that every complete local ring is Henselian. The \textit{Henselization} of a local
ring $R$ is a local ring $R^{\text{sh}}$, with the property that every local homomorphism from $R$ to a Henselian local ring factors uniquely through $R^{\text{sh}}$. The ring $R^{\text{sh}}$ is obtained by taking the direct limit of all local étale extensions $S$ of $R$ for which $(R, m) \rightarrow (S, n)$ induces an isomorphism of residue fields $R/m \cong S/n$.

A local ring $(R, m, K)$ is said to be strictly Henselian if it is Henselian and its residue field $K$ is separably closed. It is easily seen that $R$ is strictly Henselian if and only if every monic polynomial $f(T) \in R[T]$ for which $\overline{f}(T) \in K[T]$ is separable splits into linear factors in $R[T]$. Every local ring has a strict Henselization $R^{\text{sh}}$, such that every local homomorphism from $R$ to a strictly Henselian ring factors through $R^{\text{sh}}$. The strict Henselization of a field $K$ is its separable closure $K^{\text{sep}}$.

In general, the strict Henselization of a local ring $(R, m, K)$ is obtained by fixing an embedding $\iota : K \rightarrow K^{\text{sep}}$, and taking the direct limit of local étale extensions $(S, n, L)$ of $(R, m, K)$ with $L \rightarrow K^{\text{sep}}$, for which the induced map $K \rightarrow L \rightarrow K^{\text{sep}}$ agrees with $\iota : K \rightarrow K^{\text{sep}}$.

The punctured spectrum of a local ring $(R, m)$ is the set Spec $R \setminus \{m\}$, with the topology induced by the Zariski topology on Spec $R$. We say that the punctured spectrum of $R$ is formally geometrically connected if the punctured spectrum of $R^{\text{sh}}$, the completion of the strict Henselization of the completion of $R$, is connected. If $R$ is an $\mathbb{N}$-graded ring which is finitely generated over a field $R_0 = K$, then $	ext{Proj } R$ is said to be geometrically connected if $	ext{Proj} (R \otimes_K K^{\text{sep}})$ is connected.

**Definition 2.2.** Let $a$ be an ideal of a ring $R$. A ring homomorphism $\varphi : R \rightarrow S$ induces a map of local cohomology modules $H^i_a(R) \xrightarrow{\varphi_*} H^i_a(S)$. In particular, if $R$ contains a field of characteristic $p > 0$, then the Frobenius homomorphism $F : R \rightarrow R$ induces an additive map

$$H^i_a(R) \xrightarrow{F} H^i_{a^p}(R) = H^i_a(R),$$

called the Frobenius action on $H^i_a(R)$. An element $\eta \in H^i_a(R)$ is $F$-torsion if $F^e(\eta) = 0$ for some $e \in \mathbb{N}$. The module $H^i_a(R)$ is $F$-torsion if each element is $F$-torsion. The image of $F^e$ need not be an $R$-module, but it is a $K$-vector space when $K$ is perfect. In this case the $F$-stable part of $H^i_a(R)$ is the vector space

$$H^i_a(R)_{\text{st}} = \bigcap_{e \in \mathbb{N}} F^e(H^i_a(R)).$$

Some results about $F$-torsion modules and $F$-stable subspaces are summarized in Section [Ly1]. For a very general theory of $F$-modules, we refer the reader to [Ly1].

**Remark 2.3.** Consider a local ring $(R, m)$ of positive dimension. The punctured spectrum of $R$ is disconnected if and only if the minimal primes of $R$ can be partitioned into two sets $p_1, \ldots, p_m$ and $q_1, \ldots, q_n$ such that $\text{rad}(p_i + q_j) = m$ for all pairs $p_i, q_j$. Consider the graph $\Gamma$ whose vertices are the minimal primes of $R$, and there is an edge between minimal primes $p$ and $p'$ if and only if $\text{rad}(p + p') \neq m$.

It follows that the punctured spectrum of $R$ is connected if and only if the graph $\Gamma$ is connected. If the graph $\Gamma$ is connected, take a spanning tree, i.e., a connected acyclic subgraph, containing all the vertices of $\Gamma$. This spanning tree must contain a vertex $p_i$ with only one edge, so $\Gamma \setminus \{p_i\}$ is connected as well.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be incomparable prime ideals of a local domain $A$. Then their images $\bar{p}_1, \ldots, \bar{p}_n$ are precisely the minimal primes of the ring $R = A/(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n)$. From the above discussion, we conclude that if the punctured spectrum of $R$ is
connected, then there exists $i$ such that the punctured spectrum of the ring
$$A/(\mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_i \cap \cdots \cap \mathfrak{P}_n)$$
is connected as well.

Theorems 1.1 and 1.2 assert that connectedness issues for $\text{Spec } R \setminus \{m\}$ are
determined by the Frobenius action on $H^1_\mathfrak{m}(R)$. We next record an observation
about the length of $H^1_\mathfrak{m}(R)$.

**Proposition 2.4.** Let $(R, \mathfrak{m})$ be a local ring which is a homomorphic image of
a Gorenstein domain. Then $H^1_\mathfrak{m}(R)$ has finite length if and only if $\text{ann}_R \mathfrak{p} = 0$
for every prime ideal $\mathfrak{p}$ of $R$ with $\dim R/\mathfrak{p} = 1$.

**Proof.** If $\dim R = 0$, then $H^1_\mathfrak{m}(R) = 0$, and $R$ has no primes with $\dim R/\mathfrak{p} = 1$. If $\dim R = 1$, then $H^1_\mathfrak{m}(R)$ has infinite length and $\dim R/\mathfrak{p} = 1$ for some minimal prime $\mathfrak{p}$ of $R$. For the rest of the proof we hence assume that $\dim R \geq 2$.

Let $R = A/\Omega$ where $A$ is a Gorenstein domain. Localizing $A$ at the inverse
image of $\mathfrak{m}$, we may assume that $(A, \mathfrak{m})$ is a local ring. Using local duality over $A$,
the module $H^1_\mathfrak{m}(R) = H^1_\mathfrak{m}(A/\Omega)$ has finite length if and only if $\text{Ext}^\dim A^{-1}(A/\Omega, A)$
has finite length as an $A$-module. Since $\text{Ext}^\dim A^{-1}(A/\Omega, A)$ is finitely generated,
this is equivalent to the vanishing of
$$\text{Ext}^\dim A^{-1}((A/\Omega, A)_{\mathfrak{p}} = \text{Ext}^\dim A^{-1}(A_{\mathfrak{p}}/\Omega A_{\mathfrak{p}}, A_{\mathfrak{p}})$$
for all $\mathfrak{p} \in \text{Spec } A \setminus \{\mathfrak{m}\}$. Using local duality over the Gorenstein local ring
$(A_{\mathfrak{p}}, \mathfrak{P} A_{\mathfrak{p}})$, this is equivalent to the vanishing of
$$H^\dim A_{\mathfrak{p}} - \dim A + 1 (A_{\mathfrak{p}}/\Omega A_{\mathfrak{p}}) = H^\dim A_{R_{\mathfrak{p}}} - \dim A + 1 (R_{\mathfrak{p}})$$
for all $\mathfrak{p} \in \text{Spec } A \setminus \{\mathfrak{m}\}$. This local cohomology module vanishes for $\mathfrak{p} \notin \mathfrak{V}(\Omega)$. Since $\dim A_{\mathfrak{p}} - \dim A + 1 \leq 0$ for $\mathfrak{p} \in \text{Spec } A \setminus \{\mathfrak{m}\}$, we need only consider primes $\mathfrak{p} \in \mathfrak{V}(\Omega)$ with $\dim A_{\mathfrak{p}} = \dim A - 1$. Since $A$ is a catenary local domain, $\dim A_{\mathfrak{p}}$ equals $\dim A - 1$ precisely when $\dim A/\mathfrak{p} = 1$, equivalently $\dim R/\mathfrak{p} = 1$. Hence $H^1_\mathfrak{m}(R)$ has finite length if and only if $H^0_{\mathfrak{p} R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = H^0_{\mathfrak{p}}(R)$ vanishes for all $\mathfrak{p} \in \text{Spec } R$
with $\dim R/\mathfrak{p} = 1$, i.e., if and only if $\text{ann}_R \mathfrak{p} = 0$ for all $\mathfrak{p}$ with $\dim R/\mathfrak{p} = 1$. \qed

3. **Main results**

**Theorem 3.1.** Let $(R, \mathfrak{m})$ be a strictly Henselian local domain containing a field of
positive characteristic. If $R$ is a homomorphic image of a Gorenstein domain and $\dim R \geq 2$, then $H^1_\mathfrak{m}(R)$ is $F$-torsion.

**Proof.** Suppose there exists $\eta \in H^1_\mathfrak{m}(R)$ which is not $F$-torsion. Since $R$ is a domain,
Proposition 2.3 implies that $H^1_\mathfrak{m}(R)$ has finite length. Hence for all integers $e \geq 0$, the element $F^e(\eta)$ belongs to the $R$-module spanned by $\eta, F(\eta), F^2(\eta), \ldots, F^{e-1}(\eta)$. Amongst all equations of the form

$$(3.1.1) \quad F^{e+k}(\eta) + r_{1}F^{e+k-1}(\eta) + \cdots + r_{e}F^{k}(\eta) = 0$$

with $r_{i} \in R$ for all $i$, choose one where the number of nonzero coefficients $r_{i}$ that
occur is minimal. We claim that $r_{e}$ must be a unit. Note that $H^1_\mathfrak{m}(R)$ is killed by
$m^{q'}$ for some $q' = pr^e$. If $r_{e} \in \mathfrak{m}$, then applying $F^{e'}$ to equation (3.1.1), we get
$$F^{e'+e+k}(\eta) + r_{1}^{q'}F^{e'+e+k-1}(\eta) + \cdots + r_{e}^{q'}F^{e'+k}(\eta) = 0.$$
But $r \neq 0$, so this is an equation with fewer nonzero coefficients, contradicting the minimality assumption. This shows that $r_1 \in R$ is a unit. Since $\eta$ is not $F$-torsion, neither is $F^k(\eta)$, so after replacing $\eta$ if necessary, we have an equation of the form

$$F^c(\eta) + r_1 F^{c-1}(\eta) + \cdots + r_e \eta = 0$$

where $r_e$ is a unit and $\eta \in H^1_m(R)$ is not $F$-torsion. Let $\eta = [(y_1/x_1, \ldots, y_d/x_d)]$ where $H^1_m(R)$ is regarded as the cohomology of a Čech complex on a system of parameters $x_1, \ldots, x_d$ for $R$. Then (3.1.2) implies that there exists $r_{e+1} \in R$ such that each $y_i/x_i \in R_{x_i}$ is a root of the polynomial

$$f(T) = T^{p^e} + r_1 T^{p^{e-1}} + \cdots + r_e T + r_{e+1} \in R[T].$$

Now $f'(T) = r_e$ is a unit, so $\overline{f(T)} \in R/\mathfrak{m}[T]$ is a separable polynomial. Since $R$ is strictly Henselian, the polynomial $f(T)$ splits in $R[T]$, and hence any root of $f(T)$ in the fraction field of $R$ is an element of $R$. In particular, $y_1/x_1 = \cdots = y_d/x_d$ is an element of $R$, and so $\eta = 0$.

We next prove the connectedness criterion, Theorem 1.1. By Proposition 6.1, the module $H^1_m(R)$ is $F$-torsion if and only if there exists $e$ such that $F^e(H^1_m(R)) = 0$. In view of this, the following theorem is equivalent to Theorem 1.1.

**Theorem 3.2.** Let $(R, \mathfrak{m})$ be a local ring with $\dim R > 0$, which contains a field of positive characteristic. Then $H^1_m(R)$ is $F$-torsion if and only if $\dim R \geq 2$ and the punctured spectrum of $R$ is formally geometrically connected.

**Proof.** Quite generally, for a local ring $(R, \mathfrak{m})$ we have $H^1_m(\hat{R}) = H^1_m(R)$. Moreover, $S = \hat{R}^{eh}$ is a faithfully flat extension of $R$, and $H^1_m(R) \otimes_R S \cong H^i_m(S)$ is $F$-torsion if and only if $H^1_m(S)$ is $F$-torsion. Hence we may assume that $R$ is a complete local ring with a separably closed coefficient field.

Suppose that $H^1_m(R)$ is $F$-torsion. The local cohomology module $H^1_m(R)$ is not $F$-torsion by Proposition 6.1, so $\dim R \geq 2$. Let $a$ and $b$ be ideals of $R$ such that $a + b$ is $\mathfrak{m}$-primary and $a \cap b = 0$. Let

$$x_1 = y_1 + z_1, \ldots, x_d = y_d + z_d$$

be a system of parameters for $R$ where $y_i \in a$ and $z_i \in b$. Since $ab \subseteq a \cap b = 0$, we have $y_i z_j = 0$ for all $i, j$, and hence

$$y_i(y_j + z_j) = y_j(y_i + z_i).$$

These relations give an element of $H^1_m(R)$ regarded as the cohomology of a Čech complex on $x_1, \ldots, x_d$, namely

$$\eta = \left[ \begin{array}{c} y_1 \\ x_1 \\ \vdots \\ y_d \\ x_d \end{array} \right] \in H^1_m(R).$$

The hypotheses imply that $F^c(\eta) = 0$ for some $c$, so there exists $q = p^e$ and $r \in \hat{R}$ such that $(y_i/x_i)^q = r$ in $R_{x_i}$, for all $1 \leq i \leq d$. Hence there exists $t \in \mathbb{N}$ such that $x_i^t y_i^q = r x_i^{q+t}$, i.e.,

$$(y_i + z_i)^t y_i^q = r(y_i + z_i)^{q+t}.$$
and so a is \( m \)-primary. Similarly if \( 1 - r \) is a unit, then \( b \) is \( m \)-primary. This proves that the punctured spectrum of \( R \) is connected.

For the converse, assume that \( \dim R \geq 2 \) and that the punctured spectrum of \( R \) is connected. Let \( n \) denote the nilradical of \( R \). Note that \( \Spec R \) is homeomorphic to \( \Spec R/n \). Moreover, \( n \) supports a Frobenius action and is \( F \)-torsion. The long exact sequence of local cohomology relating \( H^i_m(R) \) and \( H^i_m(R/n) \) implies that if \( H^i_m(R/n) \) is \( F \)-torsion, then so is \( H^i_m(R) \), and hence there is no loss of generality in assuming that \( R \) is reduced. Let \( R = A/(\mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n) \) where \( \mathfrak{P}_1, \ldots, \mathfrak{P}_n \) are incomparable prime ideals of a power series ring \( A = K[[x_1, \ldots, x_m]] \) over a separably closed field \( K \). We use induction on \( n \) to prove that \( H^1_m(R) \) is \( F \)-torsion; the case \( n = 1 \) follows from Theorem 3.1. So we assume \( n > 1 \) below.

If \( \dim R/p_i = 1 \) for some \( i \), then \( \Spec R \setminus \{m\} \) is the disjoint union of \( V(p_i) \setminus \{m\} \) and \( V(p_1 \cap \cdots \cap p_i \cap \cdots \cap p_n) \setminus \{m\} \), contradicting the connectedness assumption. Hence \( \dim R/p_i \geq 2 \) for all \( i \). By Remark 2.4 after relabeling the minimal primes if necessary, we may assume that the punctured spectrum of \( A/\Omega \) is connected where \( \Omega = \mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_n \). The short exact sequence

\[
0 \to A/(\mathfrak{P}_1 \cap \Omega) \to A/\mathfrak{P}_1 \oplus A/\Omega \to A/(\mathfrak{P}_1 + \Omega) \to 0
\]

induces a long exact sequence of local cohomology modules containing the piece

\[
(3.2.1) \quad H^0_m(A/(\mathfrak{P}_1 + \Omega)) \to H^1_m(A/(\mathfrak{P}_1 \cap \Omega)) \to H^1_m(A/\mathfrak{P}_1) \oplus H^1_m(A/\Omega).
\]

Since \( \text{rad}(\mathfrak{P}_1 + \mathfrak{P}_i) \neq \mathfrak{M} \) for some \( i > 1 \), it follows that \( \dim A/(\mathfrak{P}_1 + \Omega) \geq 1 \). Proposition 6.2 now implies that \( H^0_m(A/(\mathfrak{P}_1 + \Omega)) \) is \( F \)-torsion. By the inductive hypothesis, \( H^1_m(A/\mathfrak{P}_1) \) and \( H^1_m(A/\Omega) \) are \( F \)-torsion as well. The exact sequence (3.2.1) implies that \( H^1_m(A/(\mathfrak{P}_1 \cap \Omega)) = H^1_m(R) \) is \( F \)-torsion.

The following lemma will be used in the proof of Theorem 1.2.

**Lemma 3.3.** Let \((R, m)\) be a complete local domain with an algebraically closed coefficient field of positive characteristic. Then \( H^1_m(R)_{\text{st}} \), the \( F \)-stable part of the module \( H^1_m(R) \), is zero.

**Proof.** If \( \dim R = 0 \), then \( H^1_m(R) = 0 \), and if \( \dim R \geq 2 \), then the assertion follows from Theorem 3.1. The remaining case is \( \dim R = 1 \). Theorem 6.3 implies that \( H^1_m(R)_{\text{st}} \) has a vector space basis \( \eta_1, \ldots, \eta_r \) such that \( F(\eta_i) = \eta_i \).

Let \( \eta \in H^1_m(R)_{\text{st}} \) be an element with \( F(\eta) = \eta \). Considering \( H^1_m(R) \) as the cohomology of a suitable Čech complex, let \( \eta \) be the class of \( y/x \) in \( R_x/R = H^1_m(R) \), where \( y \in R \) and \( x \in m \). Since \( F(\eta) = \eta \), there exists \( r \in R \) such that

\[
\left( \frac{y}{x} \right)^p - \frac{y}{x} - r = 0,
\]

and so \( y/x \in R_x \) is a root of the polynomial \( f(T) = T^p - T - r \in R[T] \). The polynomial \( f(T) \in K[T] \) is separable and \( R \) is strictly Henselian, so \( f(T) \) splits in \( R[T] \). Since \( y/x \) is a root of \( f(T) \) in the fraction field of \( R \), it must then be an element of \( R \), and hence \( \eta = 0 \). □

**Proof of Theorem 1.2.** We may assume \( R \) to be reduced by Proposition 6.5. First consider the case where the punctured spectrum of \( R \) is connected. If \( \dim R \geq 2 \), then \( H^1_m(R) \) is \( F \)-torsion by Theorem 3.2, so \( H^1_m(R)_{\text{st}} = 0 \). If \( \dim R = 1 \), then \( R \) is a domain, and Lemma 3.3 implies that \( H^1_m(R)_{\text{st}} = 0 \).

We continue by induction on the number of connected components of the punctured spectrum of \( R \). If the punctured spectrum of \( R \) is disconnected, then \( R = \ldots
$A/(\mathfrak{A} \cap \mathfrak{B})$, where $(A, \mathfrak{M})$ is a power series ring over the field $K$, and $\mathfrak{A}$ and $\mathfrak{B}$ are radical ideals of $A$ which are not $\mathfrak{M}$-primary, but $\mathfrak{A} + \mathfrak{B}$ is $\mathfrak{M}$-primary. There is a short exact sequence

$$0 \to A/(\mathfrak{A} \cap \mathfrak{B}) \to A/\mathfrak{A} \oplus A/\mathfrak{B} \to A/(\mathfrak{A} + \mathfrak{B}) \to 0.$$  

Since $H^0_{\mathfrak{M}}(A/\mathfrak{A}) = H^0_{\mathfrak{M}}(A/\mathfrak{B}) = H^1_{\mathfrak{M}}(A/(\mathfrak{A} + \mathfrak{B})) = 0$, the resulting exact sequence of local cohomology gives us

$$0 \to H^0_{\mathfrak{M}}(A/(\mathfrak{A} + \mathfrak{B})) \to H^1_{\mathfrak{M}}(A/(\mathfrak{A} \cap \mathfrak{B})) \to H^1_{\mathfrak{M}}(A/\mathfrak{A}) \oplus H^1_{\mathfrak{M}}(A/\mathfrak{B}) \to 0.$$  

By Theorem 6.4 we have a $K$-vector space isomorphism $H^1_m(R)_{\text{st}} = H^1_{\mathfrak{M}}(A/(\mathfrak{A} \cap \mathfrak{B}))_{\text{st}} \cong H^0_{\mathfrak{M}}(A/(\mathfrak{A} + \mathfrak{B}))_{\text{st}} \oplus H^1_{\mathfrak{M}}(A/\mathfrak{A})_{\text{st}} \oplus H^1_{\mathfrak{M}}(A/\mathfrak{B})_{\text{st}}$. Since $H^1_{\mathfrak{M}}(A/(\mathfrak{A} + \mathfrak{B}))_{\text{st}} = K$ by Proposition 6.2 the inductive hypothesis completes the proof. 

We next record the graded versions of the results proved in this section:

**Theorem 3.4.** Let $R$ be an $\mathbb{N}$-graded ring of positive dimension, which is finitely generated over a field $R_0 = K$ of characteristic $p > 0$.

1. If $R$ is a domain with $\dim R \geq 2$, and $K$ is separably closed, then $H^1_m(R)$ is $F$-torsion.
2. The module $H^1_m(R)$ is $F$-torsion if and only if $\dim R \geq 2$ and $\text{Proj} R$ is geometrically connected.
3. Let $K$ be a perfect field, and let $\overline{K}$ denote its algebraic closure. Then the number of connected components of $\text{Proj}(R \otimes_K \overline{K})$ is

$$1 + \dim_K H^1_m(R)_{\text{st}} = 1 + \dim_K ([H^1_m(R)]_0)_{\text{st}}.$$  

**Proof.** (1) Note that $H^1_m(R)$ is a $\mathbb{Z}$-graded $R$-module, and that

$$F: [H^1_m(R)]_n \to [H^1_m(R)]_{n+1}$$  

for all $n \in \mathbb{Z}$. The module $H^1_m(R)$ has finite length, so all elements of $H^1_m(R)$ of positive or negative degree are $F$-torsion; it remains to show that elements $\eta \in [H^1_m(R)]_0$ are $F$-torsion as well. Let $\eta$ be a element of $[H^1_m(R)]_0$ which is not $F$-torsion. As in the proof of Theorem 3.1 after a change of notation we may assume that

$$F^e(\eta) + r_1 F^{e-1}(\eta) + \cdots + r_e \eta = 0$$  

where all $r_i$ are in $[R]_0 = K$, and $r_e$ is nonzero. Let $\eta = [(y_1/x_1, \ldots, y_d/x_d)]$ where $H^1_m(R)$ is regarded as the cohomology of a homogeneous Čech complex. Then there exists $r_{e+1} \in K$ such that $y_j/x_i \in R_{x_i}$ is a root of the polynomial

$$f(T) = T^{e^e} + r_1 T^{e^{e-1}} + \cdots + r_e T + r_{e+1} \in K[T].$$  

But $f(T)$ is a separable polynomial, so it splits in $K[T]$. The element $y_i/x_i = y_j/x_j$ is a root of $f(T)$ in the fraction field of $R$, so it must be one of the roots of $f(T)$ in $K$. It follows that $\eta = 0$, which completes the proof of (1).

The proof of (2) is now similar to that of Theorem 3.2 and is left to the reader. For (3), note that $F^e(H^1_m(R))$ is a $K$-vector space since $K$ is perfect, and that

$$\dim_K H^1_m(R)_{\text{st}} = \dim_{\overline{K}} H^1_m(R \otimes_K \overline{K})_{\text{st}}.$$  

Thus we may assume $K = \overline{K}$, and the proof is similar to that of Theorem 1.2. \qed
Remark 3.5. Theorem 3.4(3) generalizes, in the case of positive characteristic, the well-known fact that the number of connected components of \(X = \text{Proj} \, R\) is
\[
\dim_K H^0(X, \mathcal{O}_X) = 1 + \dim_K [H^1_m(R)]_0,
\]
where \(R\) is an \(\mathbb{N}\)-graded reduced ring of positive dimension, which is finitely generated over an algebraically closed field \(R_0 = K\). The point is that in this case the Frobenius is bijective on \([H^1_m(R)]_0\). To see this, let
\[
\eta = \left(\frac{y_1}{x_1}, \ldots, \frac{y_d}{x_d}\right) \in [H^1_m(R)]_0
\]
be an element with \(F(\eta) = 0\), where \(H^1_m(R)\) is computed as the cohomology of a suitable Čech complex. Then there exists a homogeneous element \(r \in R\) with \((y_i/x_i)^p = r\) in \(R_x\) for all \(1 \leq i \leq d\). Such an element \(r\) must have degree zero, and hence must be an element of \(K\). But then \(r^{1/p} \in K\), and, since \(R\) is reduced, \(y_i/x_i = r^{1/p}\) for all \(i\). It follows that
\[
\eta = [(r^{1/p}, \ldots, r^{1/p})] = 0.
\]
To complete the argument, note that \([H^1_m(R)]_0\) is a finite dimensional \(K\)-vector space, and that if \(\eta_1, \ldots, \eta_n \in [H^1_m(R)]_0\) are linearly independent, then so are \(F(\eta_1), \ldots, F(\eta_n)\). It follows that \(F: [H^1_m(R)]_0 \rightarrow [H^1_m(R)]_0\) is surjective.

4. \(F\)-Purity

A ring homomorphism \(\varphi: R \rightarrow S\) is pure if \(\varphi \otimes 1: R \otimes_R M \rightarrow S \otimes_R M\) is injective for every \(R\)-module \(M\). If \(R\) is a ring containing a field of characteristic \(p > 0\), then \(R\) is \(F\)-pure if the Frobenius homomorphism \(F: R \rightarrow R\) is pure. The notion was introduced by Hochster and Roberts in the course of their study of rings of invariants in \([HR1, HR2]\).

Examples of \(F\)-pure rings include regular rings of positive characteristic and their pure subrings. If \(a\) is generated by square-free monomials in the variables \(x_1, \ldots, x_n\) and \(K\) is a field of positive characteristic, then \(K[x_1, \ldots, x_n]/a\) is \(F\)-pure.

Goto and Watanabe \([GW]\) classified one-dimensional \(F\)-pure rings: let \((R, m)\) be a local ring of positive characteristic such that \(R/m = K\) is algebraically closed, \(F: R \rightarrow R\) is finite, and \(\dim R = 1\). Then \(R\) is \(F\)-pure if and only if
\[
\check{R} \cong K[[x_1, \ldots, x_n]]/(x_i x_j \mid i < j).
\]

Two-dimensional \(F\)-pure rings have attracted a lot of attention: Watanabe \([Wat1]\) proved that \(F\)-pure normal Gorenstein local rings of dimension two are either rational double points, simple elliptic singularities, or cusp singularities. He also classified two-dimensional normal \(\mathbb{N}\)-graded rings \(R\) over an algebraically closed field \(R_0\), in terms of \(\mathbb{Q}\)-divisors on the curve \(\text{Proj} \, R\), \([Wat2]\). In \([MS]\) Mehta and Srinivas obtained a classification of two-dimensional \(F\)-pure normal singularities in terms of the resolution of the singularity. Hara completed the classification of two-dimensional normal \(F\)-pure singularities in terms of the dual graph of the minimal resolution of the singularity, \([HaN]\).

The results of Section 3 imply that over separably closed fields, \(F\)-pure domains of dimension two are Cohen-Macaulay. The point is that if \(R\) is an \(F\)-pure ring, then the Frobenius action \(F: H^1_m(R) \rightarrow H^1_m(R)\) is an injective map.
Corollary 4.1. Let $R$ be a local ring with $\dim R \geq 2$, which contains a field of positive characteristic. If $R$ is F-pure and the punctured spectrum of $R$ is formally geometrically connected, then depth $R \geq 2$.

In particular, if $R$ is a complete local F-pure domain of dimension two, with a separably closed coefficient field, then $R$ is Cohen-Macaulay.

Proof. An F-pure ring is reduced, so $H^0_m(R) = 0$. By Theorem 5.1 $H^1_m(R)$ is F-torsion. Since $R$ is F-pure, it follows that $H^1_m(R) = 0$. \hfill $\Box$

In the graded case, we similarly have:

Corollary 4.2. Let $R$ be an $\mathbb{N}$-graded ring with $\dim R \geq 2$, which is finitely generated over a field $R_0$ of positive characteristic. If $R$ is F-pure and Proj $R$ is geometrically connected, then depth $R \geq 2$.

The ring $R$ below is a graded F-pure domain of dimension two, and depth one. The issue is that Proj $R$ is connected though not geometrically connected.

Example 4.3. Let $K$ be a field of characteristic $p > 2$, and $a \in K$ an element such that $\sqrt{a} \notin K$. Let $R = K[x, y, x\sqrt{a}, y\sqrt{a}]$. The domain $R$ has a presentation

$$R = K[x, y, u, v]/(u^2 - ax^2, v^2 - ay^2, uv - axy, vx - uy),$$

and if $K^{\text{sep}}$ denotes the separable closure of $K$, then

$$R \otimes_K K^{\text{sep}} \cong K^{\text{sep}}[x, y, u, v]/(u - x\sqrt{a}, v - y\sqrt{a})(u + x\sqrt{a}, v + y\sqrt{a}).$$

Using a change of variables, $R \otimes_K K^{\text{sep}} \cong K^{\text{sep}}[x', y', u', v']/((x', y')(u', v'))$. Since $(x', y')(u', v')$ is a square-free monomial ideal, $R \otimes_K K^{\text{sep}}$ is F-pure and it follows that $R$ is F-pure. However, $R$ is not Cohen-Macaulay since $x, y$ is a homogeneous system of parameters with a non-trivial relation

$$(x\sqrt{a})y = (y\sqrt{a})x.$$ 

Using the Cech complex on $x, y$ to compute $H^1_m(R)$, we see that it is a 1-dimensional $K$-vector space generated by the element

$$\eta = \left[\frac{x\sqrt{a}}{x}, \frac{y\sqrt{a}}{y}\right] \in H^1_m(R)$$

corresponding to the relation above. Given $e \in \mathbb{N}$, let $p^e = 2k + 1$. Then

$$F^e(\eta) = a^k \eta,$$

which is a nonzero element of $H^1_m(R)$. Consequently $H^1_m(R)$ is not F-torsion, corresponding to the fact that Proj $R$ is not geometrically connected.

The corollaries obtained in this section imply that over a separably closed field, a graded or complete local F-pure domain of dimension two is Cohen-Macaulay. We record an example which shows that this is not true for rings of higher dimension.

Example 4.4. Let $K$ be a field of characteristic $p > 0$, and take

$$A = K[x_1, \ldots, x_d]/(x_1^d + \cdots + x_d^d)$$

where $d \geq 3$. Let $R$ be the Segre product of $A$ and the polynomial ring $B = K[s, t]$. Then $\dim R = d$, and the Künneth formula for local cohomology implies that

$$H^{d-1}_{m_R}(R) \cong [H^{d-1}_{m_A}(A)]_0 \otimes_K [B]_0 \cong K,$$

so $R$ is not Cohen-Macaulay. If $p \equiv 1 \pmod{d}$, then $A$ is F-pure by [HR2 Proposition 5.21]; hence $A \otimes_K B$ and its direct summand $R$ are F-pure as well.
5. Algorithmic aspects

Let $R$ be an $\mathbb{N}$-graded ring, which is finitely generated over a finite field $R_0 = K$. We wish to determine the number of geometrically connected components of the scheme $\text{Proj} R$, i.e., the number of connected components of $\text{Proj}(R \otimes_K \overline{K})$, or, equivalently, of $\text{Proj}(R \otimes_K K^{\text{sep}})$. While primary decomposition algorithms such as those of [EHV], [GTZ], or [SY], may be used to determine the connected components over the algebraic closure, $\overline{K}$. However, simply finding their number is much easier: by Theorem 3.4, this is $1 + \dim_K([H^1_m(R)]_0)_{st}$. Computing this number involves three steps.

1. Finding a good presentation of $[H^1_m(R)]_0$;
2. Determining the Frobenius action on $[H^1_m(R)]_0$ in terms of this presentation;
3. Computing the dimension of the $F$-stable part, $([H^1_m(R)]_0)_{st}$.

If $R = A/\mathfrak{A}$ for a polynomial ring $A$, we first replace $\mathfrak{A}$ by an ideal that has the same radical as $\mathfrak{A}$, but does not have the homogeneous maximal ideal $\mathfrak{M}$ as an associated prime. This can be done by saturating $\mathfrak{A}$ with respect to $\mathfrak{M}$; if desired, one may simply compute the radical of $\mathfrak{A}$, but this is often computationally expensive. Now, since $\mathfrak{M}$ is not associated to $\mathfrak{A}$, one can find a homogeneous system of parameters $x_1, \ldots, x_d$ for $R$ such that each $x_i$ is a nonzerodivisor on $R$.

The length $\ell$ of $[H^1_m(R)]_0$ may be computed by computing the length of its graded dual $[\text{Ext}_A^{n-1}(R, A(-n))]_0$, where $\dim A = n$. Of course, if this length is zero, then $X_{\overline{K}}$ is connected. Consider the Koszul cohomology modules

$$H^1(x_1^i, \ldots, x_d^i; R) = \frac{\{(a_1, \ldots, a_d) \in R^d \mid a_i x_j^i = a_j x_i^i \text{ for all } i < j\}}{\{(r x_1^i, \ldots, r x_d^i) \mid r \in R\}}.$$ 

These modules have an $\mathbb{N}$-grading, where for homogeneous elements $a_i \in R$, we define the degree of $[(a_1, \ldots, a_d)] \in H^1(x_1^i, \ldots, x_d^i; R)$ as

$$\deg[(a_1, \ldots, a_d)] = \deg a_i - \deg x_i^i,$$

which is independent of $i$. This ensures that for each $t$, the map

$$H^1(x_1^i, \ldots, x_d^i; R) \rightarrow H^1(x_1^{i+1}, \ldots, x_d^{i+1}; R)$$

$$[(a_1, \ldots, a_d)] \mapsto [(a_1 x_1^i, \ldots, a_d x_i^i)]$$

preserves degrees. The module $H^1_m(R)$ is the direct limit of these Koszul cohomology modules, and the assumption that the $x_i$ are nonzerodivisors ensures that the maps in the direct limit system are injective. The modules $H^1(x_1^1, \ldots, x_d^1; R)$ may be computed for increasing values of $t$, until we arrive at an integer $N$ such that

$$\ell\left([H^1(x_1^N, \ldots, x_d^N; R)]_0\right) = \ell.$$

This gives us a presentation for $[H^1_m(R)]_0 = [H^1(x_1^N, \ldots, x_d^N; R)]_0$, in terms of which we now analyze the Frobenius map. Replacing the $x_i$ by their powers if needed, assume that $N = 1$. Let

$$\alpha = [(a_1, \ldots, a_d)] \in [H^1(x_1, \ldots, x_d; R)]_0,$$

in which case, $F(\alpha) = [(a_1^p, \ldots, a_d^p)] \in [H^1(x_1^p, \ldots, x_d^p; R)]_0$. Since the map

$$[H^1(x_1, \ldots, x_d; R)]_0 \rightarrow [H^1(x_1^p, \ldots, x_d^p; R)]_0$$

...
Proposition 6.1. coming from the direct limit system is bijective, it follows that \( a_i^p \in x_i^{p-1}R \) for each \( 1 \leq i \leq d \). Setting \( b_i = a_i^p / x_i^{p-1} \), we arrive at

\[
F(\alpha) = \left[ (b_1, \ldots, b_d) \right] \in \left[ H^1(x_1, \ldots, x_d; R) \right]_0.
\]

Using this description of Frobenius action on the finite dimensional \( K \)-vector space \( \left[ H^1_m(R) \right]_0 = \left[ H^1(x_1, \ldots, x_d; R) \right]_0 \), it is now straightforward to compute the ranks of the vector spaces

\[
\left[ H^1_m(R) \right]_0 \supseteq F(\left[ H^1_m(R) \right]_0) \supseteq F^2(\left[ H^1_m(R) \right]_0) \supseteq \ldots,
\]

and hence of the \( F \)-stable part, \( (\left[ H^1_m(R) \right]_0)_{\text{st}} \).

6. Appendix: \( F \)-torsion modules and \( F \)-stable vector spaces

Let \( R \) be a commutative ring containing a field \( K \) of characteristic \( p > 0 \). A Frobenius action on an \( R \)-module \( M \) is an additive map \( F : M \to M \) such that \( F(rm) = r^pF(m) \) for all \( r \in R \) and \( m \in M \). In this case, \( \ker F \) is a submodule of \( M \), and we have an ascending sequence of submodules of \( M \),

\[
\ker F \subseteq \ker F^2 \subseteq \ker F^3 \subseteq \ldots.
\]

The union of these is the \( F \)-nilpotent submodule of \( M \), denoted \( M_{\text{nil}} = \bigcup_{e \in \mathbb{N}} \ker F^e \). We say \( M \) is \( F \)-torsion if \( M_{\text{nil}} = M \).

**Proposition 6.1.** Let \( (R, \mathfrak{m}) \) be a local ring containing a field of positive characteristic, and let \( M \) be an Artinian \( R \)-module with a Frobenius action. Then there exists \( e \in \mathbb{N} \) such that \( F^e(M_{\text{nil}}) = 0 \).

Hence an Artinian module \( M \) is \( F \)-torsion if and only if \( F^e(M) = 0 \) for some \( e \).

**Proof.** This is proved in [HS Proposition 1.11] under the hypothesis that \( R \) is a complete local ring with a perfect coefficient field. The general case may be concluded from this, but a more elegant approach is via Lyubeznik’s theory of \( F \)-modules; see [Ly1 Proposition 4.4].

If \( R \) is a ring containing a perfect field \( K \) of positive characteristic and \( M \) is an \( R \)-module with a Frobenius action, then \( F(M) \) is a \( K \)-vector space, and we have a descending sequence of \( K \)-vector spaces

\[
F(M) \supseteq F^2(M) \supseteq F^3(M) \supseteq \ldots.
\]

The \( F \)-stable part of \( M \) is the vector space \( M_{\text{st}} = \bigcap_{e \in \mathbb{N}} F^e(M) \).

**Proposition 6.2.** Let \( (R, \mathfrak{m}, K) \) be a local ring of dimension \( d \) which contains a field of positive characteristic.

1. \( H^0_m(R) \) is \( F \)-torsion if and only if \( d > 0 \).
2. \( H^d_m(R) \) is not \( F \)-torsion.
3. If \( d = 0 \) and \( K \) is perfect, then \( H^0_m(R)_{\text{st}} = R_{\text{st}} = K \).

**Proof.** (1) If \( d = 0 \), then \( H^0_m(R) = R \), which is not \( F \)-torsion. If \( d > 0 \), then \( H^0_m(R) \) is contained in \( \mathfrak{m} \). Since every element of \( H^0_m(R) \) is killed by a power of \( \mathfrak{m} \), it follows that each element is nilpotent. (See also [Ly2 Corollary 4.6(a)].)

(2) View \( H^d_m(R) \) as the cohomology of a Čech complex on a system of parameters \( x_1, \ldots, x_d \) for \( R \), and let \( \eta = [1 + (x_1, \ldots, x_d)] \in H^d_m(R) \). For all \( e_0 \in \mathbb{N} \), the collection of elements \( F^e(\eta) \) with \( e > e_0 \) generates \( H^d_m(R) \) as an \( R \)-module. Hence \( F^{e_0}(\eta) \) cannot be zero by Grothendieck’s nonvanishing theorem.
(3) Since \( m \) is nilpotent in this case, for integers \( e \geq 0 \) we have
\[
F^e(H_m^0(R)) = F^e(R) = \{ x^{p^e} \mid x \in R \} = \{ (y + z)^{p^e} \mid y, z \in m \} = K. \tag*{\Box}
\]

**Theorem 6.3.** Let \((R, m)\) be a local ring with a perfect coefficient field \( K \) of positive characteristic. Let \( M \) be an Artinian \( R \)-module with a Frobenius action. Then \( M_{st} \) is a finite dimensional \( K \)-vector space, and \( F : M_{st} \to M_{st} \) is an automorphism of the Abelian group \( M_{st} \).

If \( K \) is algebraically closed, then there exists a \( K \)-basis \( e_1, \ldots, e_n \) for \( M_{st} \) such that \( F(e_i) = e_i \) for all \( 1 \leq i \leq n \).

**Proof.** For the finiteness assertion, see [HS, Theorem 1.12] or [Ly1, Proposition 4.9]. It is easily seen that \( F \) is an isomorphism whenever \( M_{st} \) is finite dimensional. The existence of the special basis when \( K \) is algebraically closed follows from [Di, Proposition 5, page 233]. \( \Box \)

**Theorem 6.4 ([HS Theorem 1.13]).** Let \((R, m)\) be a complete local ring with an algebraically closed coefficient field of positive characteristic. Let \( L, M, N \) be \( R \)-modules with Frobenius actions such that we have a commutative diagram
\[
\begin{array}{ccc}
0 & \longrightarrow & L \\
\downarrow F & & \downarrow F \\
0 & \longrightarrow & L
\end{array}
\quad \begin{array}{ccc}
\longrightarrow & M & \longrightarrow & N \\
\downarrow F & & \downarrow F \\
\longrightarrow & M & \longrightarrow & N
\end{array}
\end{array}
\]
with exact rows. If \( L \) is Noetherian and \( N \) is Artinian, then the \( F \)-stable parts form a short exact sequence
\[
0 \to L_{st} \to M_{st} \to N_{st} \to 0.
\]

**Proposition 6.5.** Let \((R, m, K)\) be a complete local ring with an algebraically closed coefficient field of positive characteristic. Let \( n \) denote the nilradical of \( R \). Then for all \( i \geq 0 \), the natural map \( H^i_m(R) \to H^i_m(R/n) \), when restricted to \( F \)-stable subspaces, gives an isomorphism
\[
H^i_m(R)_{st} \cong H^i_m(R/n)_{st}.
\]

**Proof.** Let \( k \) be an integer such that \( n^{p^k} = 0 \). The short exact sequence
\[
0 \to n \to R \to R/n \to 0
\]
induces a long exact sequence of local cohomology modules
\[
\cdots \longrightarrow H^i_m(n) \overset{\alpha}{\longrightarrow} H^i_m(R) \overset{\beta}{\longrightarrow} H^i_m(R/n) \overset{\gamma}{\longrightarrow} H^{i+1}_m(n) \longrightarrow \cdots.
\]
Consider an element \( \mu \in \ker(\beta) \cap H^i_m(R)_{st} \). Then \( \mu \in \image(\alpha) \), so \( F^k(\mu) = 0 \). The Frobenius action on \( H^i_m(R)_{st} \) is an automorphism, so \( \mu = 0 \), and hence the map \( H^i_m(R)_{st} \to H^i_m(R/n)_{st} \) is injective.

To complete the proof it suffices, by Theorem 6.3, to consider an element \( \eta \in H^i_m(R/n)_{st} \) with \( F(\eta) = \eta \), and prove that it lies in the image of \( H^i_m(R)_{st} \). Now \( \gamma(\eta) \in H^{i+1}_m(n) \) so \( F^k(\gamma(\eta)) = 0 \), and therefore \( F^k(\eta) = \eta \in \ker(\gamma) \).

Let \( \eta = \beta(\mu) \) for some element \( \mu \in H^i_m(R) \). Then \( \beta(F(\mu) - \mu) = 0 \), which implies that \( F(\mu) - \mu \in \image(\alpha) \). Consequently \( F^k(F(\mu) - \mu) = 0 \), which shows that \( F^{k+1}(\mu) = F^k(\mu) \), and hence that \( F^k(\mu) \in H^i_m(R)_{st} \). Since
\[
\beta(F^k(\mu)) = F^k(\beta(\mu)) = F^k(\eta) = \eta,
\]
we are done. \( \Box \)
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