

CHARACTERISTIC p METHODS AND TIGHT CLOSURE

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These are the notes of four lectures given at the Abdus Salam International Centre for Theoretical Physics, Trieste, in June 2004 on *tight closure theory*. This theory was developed by Melvin Hochster and Craig Huneke in the paper [HH1]. An excellent account may be found in Huneke's CBMS lecture notes, [Hu]. The lectures that follow are very far from a complete treatment, but we hope they will be of some help to those who are new to the subject.

1. TIGHT CLOSURE: BASIC PROPERTIES

By a *ring* we mean a commutative ring with a unit element. A *local ring*, denoted (R, \mathfrak{m}) or (R, \mathfrak{m}, K) , is a Noetherian ring with unique maximal ideal \mathfrak{m} and residue field $K = R/\mathfrak{m}$.

If S is a subring of a ring R and there is an S -linear map $\rho : R \rightarrow S$ such that $\rho(s) = s$ for all $s \in S$, we shall say that S is a *direct summand* of R . In this situation, let \mathfrak{a} be an ideal of S , and let $s \in \mathfrak{a}R \cap S$. Then $s = \sum a_i r_i$ where $a_i \in \mathfrak{a}$ and $r_i \in R$. Applying ρ to this equation we get

$$\rho(s) = \sum a_i \rho(r_i) \in \mathfrak{a},$$

and since $\rho(s) = s$, it follows that

$$\mathfrak{a}R \cap S = \mathfrak{a}.$$

Now let G be a group acting on a Noetherian ring R . We use R^G to denote the *ring on invariants*, i.e.,

$$R^G = \{r \in R : g(r) = r \text{ for all } g \in G\}.$$

If R^G is a direct summand of R , then $\mathfrak{a}R \cap R^G = \mathfrak{a}$ for all ideals $\mathfrak{a} \subset R^G$. This has several strong consequences as we shall see in these lectures—as a start, we observe that it implies R^G is a Noetherian ring: to see this, consider a chain of ideals of the ring R^G ,

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq \dots$$

Expanding these to ideals of R , we have a chain of ideals

$$\mathfrak{a}_1 R \subseteq \mathfrak{a}_2 R \subseteq \mathfrak{a}_3 R \subseteq \dots,$$

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which stabilizes since R is Noetherian. But $\mathfrak{a}_i R \cap R^G = \mathfrak{a}_i$, so the original chain stabilizes as well.

Hilbert's fourteenth problem essentially asks whether R^G is Noetherian whenever R is. The answer turns out to be negative, and the first counterexamples were constructed by Nagata, [Na1]. We shall say more about these issues in Remark 1.8. For the moment, we focus on the case where R is a polynomial ring over a field K , and G is a finite group acting on R by K -algebra automorphisms—the subject of Benson's book [Ben]. In this case R^G is always Noetherian, see [AM, Exercise 7.5].

If the order of the finite group G is invertible in the field K , consider the map $\rho : R \rightarrow R^G$ given by

$$\rho(r) = \frac{1}{|G|} \sum_{g \in G} g(r).$$

It is easily verified that ρ is an R^G -module homomorphism, and that $\rho(s) = s$ for all $s \in R^G$. Hence R^G is a direct summand of R whenever the characteristic of K does not divide the order of the finite group G .

Example 1.1. Let S_n be the symmetric group on n symbols acting on the polynomial ring $R = K[x_1, \dots, x_n]$ by permuting the variables. Then the ring of invariants is $R^{S_n} = K[e_1, \dots, e_n]$, where e_i is the elementary symmetric function of degree i in the variables x_1, \dots, x_n . Moreover, R is a free R^{S_n} -module with basis

$$x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \quad \text{where } 0 \leq m_i \leq i - 1,$$

see, for example, [Art, Chapter II.G]. Consequently R^{S_n} is a direct summand of R , and this is independent of the characteristic of the field K .

Example 1.2. Let K be a field of characteristic other than 2. For $n \geq 3$, consider the alternating group $A_n < S_n$ acting on the polynomial ring $R = K[x_1, \dots, x_n]$ by permuting the variables. Let

$$\Delta = \prod_{i < j} (x_i - x_j) \in R.$$

Then $\sigma(\Delta) = \text{sgn}(\sigma)\Delta$ for every permutation $\sigma \in S_n$, so Δ is fixed by even cycles. It is not hard to see that $R^{A_n} = K[e_1, \dots, e_n, \Delta]$. Since Δ^2 is fixed by all elements of S_n , it must be a polynomial in the elementary symmetric functions e_i , and so R^{A_n} is a hypersurface with defining equation of the form $\Delta^2 - f(e_1, \dots, e_n) = 0$.

It turns out that R^{A_n} is a direct summand of R if and only if $|A_n| = n!/2$ is invertible in K . We examine the case $p = n = 3$ here, and refer to [Si1] or [SmL] for details of the general case. The ring of invariants is

$$R^{A_3} = K[e_1, e_2, e_3, \Delta]$$

where $e_1 = x_1 + x_2 + x_3$, $e_2 = x_1x_2 + x_2x_3 + x_3x_1$, $e_3 = x_1x_2x_3$, and $\Delta = (x_1 - x_2)(x_2 - x_3)(x_1 - x_3)$. Since R^{A_3} is a hypersurface with defining equation $\Delta^2 - f(e_1, e_2, e_3) = 0$, it follows that $\Delta \notin (e_1, e_2, e_3)R^{A_3}$. On the other hand

$$\Delta = (x_1 - x_2)(x_3e_1 + e_2) \in (e_1, e_2, e_3)R,$$

so R^{A_3} is not a direct summand of R .

For a finite group G , when is R^G a direct summand of R ? One answer comes from tight closure theory:

Theorem 1.3. *Let K be a field of positive characteristic, and let G be a finite group acting on a polynomial ring $R = K[x_1, \dots, x_n]$ by degree preserving K -algebra automorphisms. Then R^G is a direct summand of R if and only if R^G is weakly F -regular.*

Definition 1.4. Let R be a ring of prime characteristic p and let R° denote the complement of the minimal primes of R . If R is a domain, which will be the main case in these lectures, then R° is just the set of nonzero elements of R . For an ideal $\mathfrak{a} = (x_1, \dots, x_n)$ of R and a prime power $q = p^e$, we use the notation $\mathfrak{a}^{[q]} = (x_1^q, \dots, x_n^q)$. The *tight closure* of \mathfrak{a} is

$$\mathfrak{a}^* = \{z \in R \mid \text{there exists } c \in R^\circ \text{ for which } cz^q \in \mathfrak{a}^{[q]} \text{ for all } q \gg 0\}$$

which is an ideal of R containing \mathfrak{a} —possibly larger, but a tight fit nonetheless. A ring R is *weakly F -regular* if $\mathfrak{a}^* = \mathfrak{a}$ for all ideals \mathfrak{a} of R .

For a ring R of characteristic $p > 0$, the map $F : R \rightarrow R$ with $F(r) = r^p$ is a ring homomorphism, called the *Frobenius* homomorphism. Note that R may be viewed as an R -module via the Frobenius homomorphism or any iteration thereof. For an ideal $\mathfrak{a} = (x_1, \dots, x_n)$ of R , consider the exact sequence

$$R^n \xrightarrow{(x_1 \ \cdots \ x_n)} R \longrightarrow R/\mathfrak{a} \longrightarrow 0.$$

Applying $F^e(R) \otimes_R$, the right exactness of tensor gives us the exact sequence

$$R^n \xrightarrow{(x_1^q \ \cdots \ x_n^q)} R \longrightarrow F^e(R) \otimes_R R/\mathfrak{a} \longrightarrow 0,$$

which shows that $F^e(R) \otimes_R R/\mathfrak{a} \cong R/\mathfrak{a}^{[q]}$.

If $R = \mathbb{Z}/p[x_1, \dots, x_d]$ is a polynomial ring, the Frobenius $F : R \rightarrow R$ may be identified with the inclusion

$$\mathbb{Z}/p[x_1^p, \dots, x_d^p] \subset \mathbb{Z}/p[x_1, \dots, x_d]. \quad (\#)$$

The monomials in x_i with each exponents less than p form a basis for $\mathbb{Z}/p[x_1, \dots, x_d]$ as a $\mathbb{Z}/p[x_1^p, \dots, x_d^p]$ -module, hence the inclusion $(\#)$ is free, in particular, flat. More generally, we have:

Proposition 1.5. *Let R be a regular ring of prime characteristic p . Then the Frobenius homomorphism is flat.*

Proof. The issue is local, so we may assume R is a regular local ring. It suffices to verify the assertion after taking the completion of R at its maximal ideal, so by the structure theorem for complete local rings, Theorem 5.1 of the Appendix, we may assume R is a power series ring over a field, say $R = K[[x_1, \dots, x_d]]$. By flat descent we reduce to the case $K = \mathbb{Z}/p$ and then, similar to the argument earlier, the Frobenius $F : R \rightarrow R$ may be identified with the inclusion $\mathbb{Z}/p[[x_1^p, \dots, x_d^p]] \subset \mathbb{Z}/p[[x_1, \dots, x_d]]$, which is free, hence flat. \square

The converse is a theorem of Kunz: a ring R of positive characteristic is regular if and only if the Frobenius homomorphism $F : R \rightarrow R$ is flat, [Ku1, Her].

Note that for a finite group G acting on a ring R , the extension $R^G \subseteq R$ is integral since an element $r \in R$ is a root of the polynomial

$$\prod_{g \in G} (x - g(r)) \in R^G[x].$$

The proof of Theorem 1.3 will be immediate from the three fundamental properties of weakly F -regular rings which we establish next.

Theorem 1.6. *The following are true for rings of positive characteristic:*

- (1) *Regular rings are weakly F -regular.*
- (2) *Direct summands of weakly F -regular domains are weakly F -regular.*
- (3) *An excellent weakly F -regular domain is a direct summand of every module-finite extension domain.*

Proof. (1) The key point is the flatness of the Frobenius homomorphism, Proposition 1.5. Let R be a regular ring of characteristic $p > 0$. Given an ideal \mathfrak{a} of R and an element $z \in R$, there is a short exact sequence

$$0 \rightarrow R/(\mathfrak{a} : z) \rightarrow R/\mathfrak{a} \rightarrow R/(\mathfrak{a} + zR) \rightarrow 0.$$

Tensoring this with an iteration $F^e : R \rightarrow R$ of the Frobenius, and using that F^e is flat, we get a short exact sequence

$$0 \rightarrow R/(\mathfrak{a} : z)^{[q]} \rightarrow R/\mathfrak{a}^{[q]} \rightarrow R/(\mathfrak{a} + zR)^{[q]} \rightarrow 0,$$

which implies that $\mathfrak{a}^{[q]} : z^q = (\mathfrak{a} : z)^{[q]}$ for all $q = p^e$. If $z \in \mathfrak{a}^*$ then, by definition, there exists $c \in R^\circ$ such that $cz^q \in \mathfrak{a}^{[q]}$ for all $q \gg 0$. But then

$$c \in \mathfrak{a}^{[q]} : z^q = (\mathfrak{a} : z)^{[q]} \quad \text{for all } q \gg 0.$$

If $z \notin \mathfrak{a}$, there exists a maximal ideal \mathfrak{m} of R with $\mathfrak{a} : z \subseteq \mathfrak{m}$. We then get

$$c \in \bigcap_{q \gg 0} (\mathfrak{a} : z)^{[q]} R_{\mathfrak{m}} \subseteq \bigcap_{q \gg 0} (\mathfrak{m} R_{\mathfrak{m}})^{[q]},$$

but then $c \in \bigcap_{q \gg 0} (\mathfrak{m}R_{\mathfrak{m}})^q$ which is zero by Krull's intersection theorem. This contradicts the assumption that $c \in R^\circ$.

(2) Let S be a direct summand of a weakly F-regular domain R . For an ideal $\mathfrak{a} \subset S$ and element $z \in S$, suppose that $z \in \mathfrak{a}^*$. Then $cz^q \in \mathfrak{a}^{[q]}$ for all $q \gg 0$, where $c \in S$ is a nonzero element. This implies that $cz^q \in \mathfrak{a}^{[q]}R$ for all $q \gg 0$, and so $z \in (\mathfrak{a}R)^* = \mathfrak{a}R$. Since S is a direct summand of R we have $\mathfrak{a}R \cap S = \mathfrak{a}$, and hence we get $z \in \mathfrak{a}$.

(3) Let $S \subseteq R$ be a module-finite extension of domains. Then the fraction field $Q(R)$ of R is a finite dimensional vector space over the fraction field $Q(S)$ of S . Choose an $Q(S)$ -linear map $\varphi_0 : Q(R) \rightarrow Q(S)$ with $\varphi_0(1) \neq 0$. Since $\varphi_0(R)$ is a finitely generated S -submodule of $Q(S)$, there exists a nonzero element $d \in S$ such that $\varphi = d\varphi_0 : R \rightarrow S$. Let $c = \varphi(1) \in S$ which, we note, is a nonzero element.

Let $\mathfrak{a} = (x_1, \dots, x_n)$ be an ideal of S , and let $z \in \mathfrak{a}R \cap S$. Then there exist elements $r_i \in R$ such that

$$z = r_1x_1 + \dots + r_nx_n.$$

Taking Frobenius powers of this equation, we get

$$z^q = r_1^qx_1^q + \dots + r_n^qx_n^q \quad \text{for all } q = p^e.$$

Applying φ now gives us

$$cz^q = \varphi(1 \cdot z^q) = \varphi(r_1^q)x_1^q + \dots + \varphi(r_n^q)x_n^q \in \mathfrak{a}^{[q]} \quad \text{for all } q = p^e,$$

and so $z \in \mathfrak{a}^*$. What we have proved is that if $S \subseteq R$ is a module-finite extension of domains of positive characteristic, then $\mathfrak{a}R \cap S \subseteq \mathfrak{a}^*$ for all ideals \mathfrak{a} of S . If S is weakly F-regular then, for all ideals \mathfrak{a} of S , we get $\mathfrak{a}R \cap S = \mathfrak{a}$. To complete the proof, we need a result of Hochster [Ho2]: given a module-finite extension $S \subseteq R$ of excellent domains, S is a direct summand of R if and only if $\mathfrak{a}R \cap S = \mathfrak{a}$ for all ideals $\mathfrak{a} \subset S$, see Theorem 5.10. \square

Remark 1.7. We saw that excellent weakly F-regular domains are direct summands of module-finite extension domains, Theorem 1.6 (3). An integral domain is said to be a *splinter* if it is a direct summand of every module-finite extension domain. It is easy to verify that a splinter is a normal domain.

(i) Characteristic zero: If a normal domain S contains the field of rational numbers and R is a module-finite extension domain, then the trace map of fraction fields $Q(R) \rightarrow Q(S)$ can be used to construct a splitting $\rho : R \rightarrow S$. Consequently an integral domain of characteristic zero is a splinter if and only if it is normal.

(ii) Mixed characteristic: It is for rings of mixed characteristic that the local homological conjectures remain unresolved. In this case, *the canonical element conjecture*,

the improved new intersection conjecture, and the monomial conjecture are equivalent to the conjecture that every regular local ring is a splinter, which is known as the direct summand conjecture. For work on these and related homological questions, we refer the reader to the papers [Du, EG, Hei, Ho1, Ho3, PS, Ro1, Ro2].

(iii) Positive characteristic: As we saw, excellent weakly F-regular domains of positive characteristic are splinters, and Hochster and Huneke also proved the converse for Gorenstein rings, [HH4]. This was extended by the author to the class of \mathbb{Q} -Gorenstein rings in [Si2] and, more recently, to rings whose anti-canonical cover is Noetherian. One of the incentives for proving that the splinter property and weak F-regularity agree for rings of positive characteristic is that it is easy to show the localization of a splinter is a splinter. It is an open question whether weak F-regularity localizes in general, though this is known in the graded case, [LyS]. This explains the nomenclature: the term *F-regular* is reserved for rings for which every localization is weakly F-regular.

These issues are closely related to the question whether the tight closure \mathfrak{a}^* of an ideal \mathfrak{a} agrees with its plus closure, $\mathfrak{a}^+ = \mathfrak{a}R^+ \cap R$, where R^+ denotes the integral closure of R in an algebraic closure of its fraction field. (An excellent domain R is a splinter if and only if $\mathfrak{a}^+ = \mathfrak{a}$ for all ideals \mathfrak{a} of R .) The containment $\mathfrak{a}^+ \subseteq \mathfrak{a}^*$ is easily seen, and Smith established the equality $\mathfrak{a}^+ = \mathfrak{a}^*$ for parameter ideals (see Definition 2.12) of excellent domains, [Sm1]. Whether tight closure and plus closure agree is perhaps the most fascinating problem in tight closure theory. Work on this question eventually led Hochster and Huneke to the celebrated theorem that R^+ is a big Cohen-Macaulay algebra for any excellent local domain R of positive characteristic, [HH2]. A related result is that the separable part of R^+ is also a big Cohen-Macaulay algebra, [Si3]. It should be mentioned that amongst various other consequences, establishing the equality of \mathfrak{a}^* and \mathfrak{a}^+ would prove that tight closure localizes, a problem that has persisted since the inception of tight closure theory.

Remark 1.8. A linear algebraic group is Zariski closed subgroup of a general linear group $GL_n(K)$. A linear algebraic group G is *linearly reductive* if every finite dimensional G -module is a direct sum of irreducible G -modules, equivalently, if every G -submodule has a G -stable complement.

Linearly reductive groups in characteristic zero include finite groups, algebraic tori (i.e., products of copies of the multiplicative group of the field), and the classical groups $GL_n(K)$, $SL_n(K)$, $Sp_{2n}(K)$, $O_n(K)$ and $SO_n(K)$.

A linear algebraic group is *reductive* if its largest closed connected solvable normal subgroup is an algebraic torus. In characteristic zero, linearly reductive groups are precisely those which are reductive.

If a linearly reductive group acts on a finitely generated K -algebra R , say by degree preserving K -algebra automorphisms, then there is an R^G -linear map, the *Reynolds operator* $\rho : R \rightarrow R^G$, which makes R^G a direct summand of R . By our earlier discussion, it then follows that R^G is Noetherian. However, if the field K has positive characteristic, there need not be a Reynolds operator—reductive groups in positive characteristic usually fail to be linearly reductive, Example 4.6. In the preface of his book [Mum], Mumford conjectured that reductive groups satisfy a weaker property which should ensure that R^G is Noetherian, and this led to the notion of geometrically reductive groups. A linear algebraic group G is *geometrically reductive* if for every finite dimensional G -module V , and G -stable submodule W of codimension one such that G acts trivially on V/W , there exists $n \in \mathbb{N}$ such that $W \cdot S^n(V)$ has a G -stable complement in $S^n(V)$.

In [Na3] Nagata proved that R^G is finitely generated if G is geometrically reductive and Haboush, in [Hab], settled Mumford's conjecture by proving that reductive groups are geometrically reductive. It is interesting to note that for reductive groups G , though $\mathfrak{a}R \cap R^G$ may not be contained in \mathfrak{a} , we always have $\mathfrak{a}R \cap R^G \subseteq \text{rad}(\mathfrak{a})$, [Na3, Lemma 5.2.B].

2. THE COHEN-MACAULAY PROPERTY

Definition 2.1. Let M be an R -module. Elements z_1, \dots, z_d of R form a *regular sequence* on M if

- (1) $(z_1, \dots, z_d)M \neq M$, and
- (2) z_i is not a zerodivisor on $M/(z_1, \dots, z_{i-1})M$ for every i with $1 \leq i \leq d$.

A local ring (R, \mathfrak{m}) is *Cohen-Macaulay* if some (equivalently, every) system of parameters for R is a *regular sequence* on R . A ring R is *Cohen-Macaulay* if the local ring $R_{\mathfrak{m}}$ is Cohen-Macaulay for every maximal ideal \mathfrak{m} of R .

If R is an \mathbb{N} -graded ring finitely generated over a field $R_0 = K$, then R is Cohen-Macaulay if and only if some (equivalently, every) homogeneous system of parameters for R is a regular sequence.

For a local (or \mathbb{N} -graded) ring (R, \mathfrak{m}) , and a (graded) R -module M , the *depth* of M , denoted $\text{depth } M$, is the length of a maximal sequence of elements of \mathfrak{m} which form a regular sequence on M . Consequently a local ring R is Cohen-Macaulay if and only if $\text{depth } R = \dim R$.

How does the Cohen-Macaulay property arise in invariant theory? We start with an elementary example:

Example 2.2. Let K be an infinite field, and $R = K[x_1, x_2, y_1, y_2]$. Consider the action of the multiplicative group $G = K \setminus \{0\}$, as follows:

$$\lambda \in G : f(x_1, x_2, y_1, y_2) \mapsto f(\lambda x_1, \lambda x_2, \lambda^{-1} y_1, \lambda^{-1} y_2).$$

Note that under this action, every monomial is taken to a scalar multiple. Let $f \in R$ be a polynomial which is fixed by the group action. If a monomial $x_1^i x_2^j y_1^k y_2^l$ occurs in f with nonzero coefficient, comparing coefficients of this monomial in f and $\lambda(f)$ gives us

$$\lambda^{i+j-k-l} = 1 \quad \text{for all } \lambda \in G.$$

Since G is infinite, we must have $i + j = k + l$. It follows that the ring of invariants is precisely

$$R^G = K[x_1 y_1, x_1 y_2, x_2 y_1, x_2 y_2].$$

Note that $\dim R^G = 3$, for example, by examining the transcendental degree of the fraction field. The polynomial ring $S = K[z_{11}, z_{12}, z_{21}, z_{22}]$ surjects onto R^G via the K -algebra homomorphism φ with $\varphi : z_{ij} \mapsto x_i y_j$. It is easily seen that

$$\varphi(z_{11} z_{22} - z_{12} z_{21}) = 0.$$

Since $\dim S = 4$, the kernel of φ must be a height one prime of S , and it follows that $\ker \varphi = (z_{11} z_{22} - z_{12} z_{21})$.

Remark 2.3. Given an action of G on a polynomial ring R , the *first fundamental problem of invariant theory*, according to Hermann Weyl [We], is to find generators for the ring of invariants R^G , in other words to find a polynomial ring S with a surjection $\varphi : S \rightarrow R^G$. The *second fundamental problem* is to find relations amongst these generators, i.e., to find a free S -module S^{b_1} which surjects onto $\ker \varphi$. In Example 2.2, we solved these two fundamental problems for the prescribed group action. In general, continuing this sequence of fundamental questions, one would like to determine the *resolution* of R^G as an S -module, i.e., to determine an exact complex

$$\dots \rightarrow S^{b_3} \rightarrow S^{b_2} \rightarrow S^{b_1} \rightarrow S \xrightarrow{\varphi} R^G \rightarrow 0.$$

Hilbert's syzygy theorem implies that a minimal such resolution is finite. (Since R^G is a graded module over the polynomial ring S , *minimal* can be taken to mean that the entries of the matrices defining the maps $S^{b_{i+1}} \rightarrow S^{b_i}$ are homogeneous elements of S which are non-units.) Knowing the graded free modules occurring in the resolution, it is then easy to compute the dimension, multiplicity and, more generally, the Hilbert polynomial of R^G . Another fundamental question then arises: what is the length of the minimal resolution of R^G as an S -module, i.e., what is the *projective dimension* $\text{pd}_S R^G$? By the Auslander-Buchsbaum formula,

$$\text{pd}_S R^G = \text{depth } S - \text{depth } R^G.$$

The polynomial ring S is Cohen-Macaulay and $\text{depth } R^G \leq \dim R^G$, so we get a lower bound for the length of a minimal resolution,

$$\text{pd}_S R^G \geq \dim S - \dim R^G.$$

Equality holds if and only if $\text{depth } R^G = \dim R^G$, i.e., precisely when R^G is Cohen-Macaulay. This leads us to the question: *When is R^G Cohen-Macaulay?*

Theorem 2.4 (Hochster-Eagon, [HE]). *Let R be a polynomial ring over a field K , and let G be a finite group acting on R by degree preserving K -algebra automorphisms. If R^G is a direct summand of R , then R^G is Cohen-Macaulay. In particular, if $|G|$ is invertible in K , then R^G is Cohen-Macaulay.*

Proof. Let $\underline{z} = z_1, \dots, z_d$ be a homogeneous system of parameters for the ring R^G . Since G is finite, R is an integral extension of R^G . Consequently \underline{z} is a system of parameters for R , and hence is a regular sequence on R . But R^G is a direct summand of R , so \underline{z} is a regular sequence on R^G as well. \square

Remark 2.5. There are examples of finite groups G for which R^G is not Cohen-Macaulay due to Bertin and Fossum-Griffith: Let $R = K[x_1, \dots, x_q]$ be a polynomial ring over a field K of characteristic $p > 0$ where $q = p^e$. Fix a generator σ for the cyclic group $G = \mathbb{Z}/q$, and consider the K -linear action of G on R where σ cyclically permutes the generators x_1, \dots, x_q . Then for $q \geq 4$, the ring of invariants R^G is a unique factorization domain which is not Cohen-Macaulay. Moreover, this is preserved if R is replaced by its completion \widehat{R} at the homogeneous maximal ideal, and the action of G on \widehat{R} is the unique continuous action extending the one on R . For proofs, see [Ber] and [FG].

The proof of Theorem 2.4 works more generally to show that a direct summand S of a Cohen-Macaulay ring R is Cohen-Macaulay provided that a system of parameters for S forms part of a system of parameters for R . In general, a direct summand of a Cohen-Macaulay ring need not be Cohen-Macaulay as we see in the next example.

Example 2.6. Let K be an infinite field, and let R be the hypersurface

$$R = K[x, y, z, s, t]/(x^3 + y^3 + z^3).$$

Then R has a K -linear action of $G = K \setminus \{0\}$ where

$$\lambda : \begin{cases} x \mapsto \lambda x \\ y \mapsto \lambda y \\ z \mapsto \lambda z \end{cases} \quad \text{and} \quad \lambda : \begin{cases} s \mapsto \lambda^{-1} s \\ t \mapsto \lambda^{-1} t \end{cases} \quad \text{for } \lambda \in G.$$

Similar to Example 2.2, the ring of invariants R^G is the K -algebra generated by the elements sx, sy, sz, tx, ty , and tz . It is easy to see that R^G is a direct summand of the Cohen-Macaulay ring R . (More generally, an algebraic torus is linearly reductive.) However R^G is not Cohen-Macaulay: the elements $sx, ty, sy - tx$

form a homogeneous system of parameters for R^G and satisfy the relation

$$sztz(sy - tx) = (sz)^2ty - (tz)^2sx,$$

and therefore $sy - tx$ is a zerodivisor on $R^G/(sx, ty)R^G$.

Though the Cohen-Macaulay property is not preserved by direct summands, there is a beautiful theorem of Hochster and Roberts which implies that several important rings of invariants are Cohen-Macaulay:

Theorem 2.7 ([HR]). *Let G be a linearly reductive group acting linearly on a polynomial ring R . Then R^G is Cohen-Macaulay. More generally, a direct summand of a polynomial ring is Cohen-Macaulay.*

This has been extended by Hochster and Huneke to all equicharacteristic regular rings using their construction of big Cohen-Macaulay algebras:

Theorem 2.8 ([HH5]). *Let S be a direct summand of a regular ring containing a field. Then S is Cohen-Macaulay.*

We have seen that direct summands of weakly F-regular domains are weakly F-regular, Theorem 1.6 (2). In view of this, the following theorem gives us an elementary proof of the Hochster-Roberts theorem in the positive characteristic graded case. We will return to the characteristic zero case in § 4.

Theorem 2.9. *Let (R, \mathfrak{m}, K) be a complete local domain of characteristic $p > 0$ or an \mathbb{N} -graded domain finitely generated over a field $[R]_0 = K$. Let y_1, \dots, y_d be a system of parameters for R , or a homogeneous system of parameters in the graded case. Then*

$$(y_1, \dots, y_i) : y_{i+1} \subseteq (y_1, \dots, y_i)^* \quad \text{for all } 0 \leq i \leq d-1.$$

In particular, if R is weakly F-regular, then it is Cohen-Macaulay.

Proof. In the complete case, R is module-finite over $A = K[[y_1, \dots, y_d]]$, and in the graded case over the graded subring $A = K[y_1, \dots, y_d]$. Suppose there exist $z, r_1, \dots, r_i \in R$ with

$$zy_{i+1} = r_1y_1 + \dots + r_iy_i,$$

taking p^e th powers, we get

$$z^q y_{i+1}^q = r_1^q y_1^q + \dots + r_i^q y_i^q \quad \text{for all } q = p^e. \quad (\#)$$

Let A^t be a free A -submodule of R where t is as large as possible. Then R/A^t is a finitely generated torsion A -module, hence is killed by some nonzero element $c \in A$. Since $cR \subseteq A^t$, multiplying the equation (#) by c , we get

$$cz^q y_{i+1}^q \in (y_1^q, \dots, y_i^q)A^t.$$

The elements y_1, \dots, y_{i+1} form a regular sequence on A , hence on the free A -module A^t , and so we get

$$cz^q \in (y_1^q, \dots, y_i^q)A^t \subseteq (y_1^q, \dots, y_i^q)R \quad \text{for all } q = p^e.$$

This implies that $z \in (y_1, \dots, y_i)^*$ as desired. \square

Under mild hypotheses, weak F-regularity is preserved on taking direct summands and implies the Cohen-Macaulay property. There is another class of rings in characteristic zero with these properties:

Definition 2.10. Let X be a normal irreducible variety over an algebraically closed field K of characteristic zero. Then X has *rational singularities* if for some (equivalently, every) desingularization $f : Z \rightarrow X$, we have $R^i f_* (\mathcal{O}_Z) = 0$ for all $i \geq 1$. If X has rational singularities, then all local rings of X are Cohen-Macaulay. We say that R has rational singularities if $\text{Spec } R$ has rational singularities. There are useful numerical criterion to detect when a graded ring has rational singularities, see [F] or [Wa2].

Theorem 2.11 (Boutot, [Bo]). *Let R be a finitely generated algebra over a field of characteristic zero, and S be a direct summand of R . If R has rational singularities, then so does S .*

The rational singularity property is related to a property which arise in tight closure theory:

Definition 2.12. An ideal $\mathfrak{a} = (x_1, \dots, x_n)$ is said to be a *parameter ideal* if the images of x_1, \dots, x_n form part of a system of parameters in the local ring $R_{\mathfrak{p}}$ for every prime ideal \mathfrak{p} containing \mathfrak{a} . A ring R of positive characteristic is *F-rational* if every parameter ideal of R equals its tight closure.

Of course, a weakly F-regular ring is F-rational. The notions agree for Gorenstein rings, but not in general, see [Wa3].

A ring $R = K[x_1, \dots, x_n]/\mathfrak{a}$ finitely generated over a field K of characteristic zero is said to be of *dense F-rational type* if there exists a finitely generated \mathbb{Z} -algebra $A \subset K$ and a finitely generated free A -algebra

$$R_A = A[x_1, \dots, x_n]/\mathfrak{a}_A$$

such that $R \cong R_A \otimes_A K$ and, for all maximal ideals \mathfrak{m} in a Zariski dense subset of $\text{Spec } A$, the fiber rings $R_A \otimes_A A/\mathfrak{m}$ are F-rational rings of characteristic $p > 0$.

Smith proved that rings of dense F-rational type have rational singularities [Sm2], and the converse is a theorem of Hara [Har] as well as Mehta-Srinivas [MS]. Combining these results, we have:

Theorem 2.13. *Let R be a ring finitely generated over a field of characteristic 0. Then R has rational singularities if and only if it is of dense F -rational type.*

For other striking connections between tight closure theory and singularities in characteristic zero see [HW]. We conclude this section with some examples of rings of invariants. Although we will not prove anything (except perhaps in one case) and refer the reader to [We] and [DP], we hope the following examples give a glimpse of this rich subject.

Example 2.14. Let $X = (x_{ij})$ be an $n \times d$ matrix of variables over a field K , and consider the polynomial ring $R = K[X]$, i.e., R is a polynomial ring in nd variables. Let $G = SL_n(K)$ be the special linear group acting on R as follows:

$$M : x_{ij} \longrightarrow (MX)_{ij},$$

i.e., an element $M \in G$ send x_{ij} , the (i, j) entry of the matrix X , to the (i, j) entry of the matrix MX . Since $\det M = 1$, it follows that the size n minors of X are fixed by the group action. It turns out whenever K is infinite, R^G is the K -algebra generated by these size n minors. The ring R^G is the homogeneous coordinate ring of the Grassmann variety of n dimensional subspaces of a d -dimensional vector space. The relations between the minors are the well-known Plücker relations.

The reader is invited to prove that R^G is a unique factorization domain. (The key point is that since the commutator of the group $G = SL_n(K)$ is $[G, G] = G$, any homomorphism from G to an abelian group must be trivial. In particular, there are no nontrivial homomorphisms from G to the multiplicative group of the field K .) Once we know that R^G is a unique factorization domain, Murthy's theorem [Mur] implies that R^G is Gorenstein. More generally, the ring of invariants of a connected semisimple linear algebraic group acting linearly on a polynomial ring is a Cohen-Macaulay unique factorization domain, hence also Gorenstein. For more on the Gorenstein property of R^G see [Wa1] and [Kn].

Example 2.15. Let $X = (x_{ij})$ and $Y = (y_{jk})$ be $r \times n$ and $n \times s$ matrices of variables over an infinite field K , and consider the polynomial ring $R = K[X, Y]$ of dimension $rn + ns$. Let $G = GL_n(K)$ be the general linear group acting on R where $M \in G$ maps the entries of X to corresponding entries of XM^{-1} and the entries of Y to those of MY . Then R^G is the K -algebra generated by the entries of the product matrix XY . If $Z = (z_{ij})$ is an $r \times s$ matrix of new variables mapping onto the entries of XY , the kernel of the induced K -algebra surjection $K[Z] \longrightarrow R^G$ is the ideal generated by the size $n + 1$ minors of the matrix Z . These *determinantal rings* are the subject of [BV]. The case where $r = s = 2$ and $n = 1$ was earlier encountered in Example 2.2.

Let us finally go through a computation in some detail:

Example 2.16. Let $X = (x_{ij})$ be an $n \times n$ matrix of variables over an infinite field K , and consider the polynomial ring $R = K[X]$. Let $G = GL_n(K)$ be the general linear group acting linearly on R where $M \in G$ maps entries of the matrix X to corresponding entries of MXM^{-1} . We shall determine the ring of invariants R^G . This example is a special case of [Pr].

The matrices X and MXM^{-1} are conjugate, so $\det(X)$, $\text{trace}(X)$, and, more generally, the coefficients of the characteristic polynomial $p(t) = \det(tI - X)$ of X are fixed by the group action. We claim that R^G is the K -algebra generated by the coefficients of $p(t)$.

Let $Y = (y_{ij})$ be an $n \times n$ matrix of new variables, and set $S = R[Y, 1/\det(Y)]$. Given $f(X) \in R^G$, consider the element $f(YXY^{-1}) \in S$. When Y is specialized to any matrix in $GL_n(K)$, the specialization of $f(YXY^{-1})$ agrees with $f(X)$. Since $f(YXY^{-1}) - f(X)$ vanishes for all such specializations, it must vanish identically, i.e., $f(YXY^{-1}) = f(X)$.

Let L be the algebraic closure of the fraction field of R . When we specialize the off-diagonal entries of X to 0, the resulting matrix has distinct eigenvalues x_{11}, \dots, x_{nn} , and it follows that X has distinct eigenvalues in L . Consequently there exists a matrix $N \in GL_n(L)$ such that $NXN^{-1} = D$ is diagonal, the entries of D being the eigenvalues of X . Specializing Y to the matrix N , we see that $f(D) = f(X)$. Hence $f(X)$ is a polynomial in the entries of D , i.e., $f(X)$ is a polynomial function of the eigenvalues of X . Moreover, for a permutation $\pi \in S_n$, consider the corresponding permutation matrix P . Then $f(PDP^{-1}) = f(D)$, so $f(X)$ is a symmetric function of the eigenvalues of $f(X)$. The elementary symmetric functions of the eigenvalues are, up to sign, the coefficients of the characteristic polynomial, so we have proved our claim.

We have now solved the first fundamental problem for this group action. The second fundamental problem is to determine the relations, if any, between the coefficients of the characteristic polynomial of X . Specializing the off-diagonal entries of X to 0, the coefficients of the characteristic polynomial of the resulting matrix are the n elementary symmetric functions in x_{11}, \dots, x_{nn} which are algebraically independent. It follows that the coefficients of $p(t)$ are algebraically independent as well, so R^G is a polynomial ring in n variables.

3. THE BRIANÇON-SKODA THEOREM

Definition 3.1. Let \mathfrak{a} be an ideal of a ring R . An element $z \in R$ is in the *integral closure* of \mathfrak{a} , denoted $\bar{\mathfrak{a}}$, if it satisfies an equation of the form

$$z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n = 0$$

where $a_i \in \mathfrak{a}^i$ for all $1 \leq i \leq n$. If R is Noetherian, another characterization of the integral closure is that $z \in \bar{\mathfrak{a}}$ if and only if there exists $c \in R^\circ$ such that $cz^n \in \mathfrak{a}^n$ for infinitely many positive integers n , or equivalently, for all $n \gg 0$. Yet another characterization in the Noetherian case is that $z \in \bar{\mathfrak{a}}$ if and only if $\varphi(z) \in \mathfrak{a}V$ for every homomorphism $\varphi : R \rightarrow V$, where V is a rank one discrete valuation ring.

It is easy to see that $\bar{\mathfrak{a}}$ is an ideal of R with

$$\mathfrak{a} \subseteq \bar{\mathfrak{a}} \subseteq \text{rad}(\mathfrak{a}).$$

Moreover, if R has characteristic $p > 0$, then $\mathfrak{a}^* \subseteq \bar{\mathfrak{a}}$.

Let $R = \mathbb{C}\{x_1, \dots, x_d\}$ be the ring of convergent power series in d variables over the complex numbers. If f belongs to the maximal ideal of R , then, using the valuation criterion, we can see that

$$f \in \overline{\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_d} \right)}.$$

This implies that some power f^k of f belongs to the ideal generated by the partial derivatives, and John Mather asked if there is a bound on this power k , [Wal]. If f is a homogeneous polynomial of degree n , then $k = 1$ suffices since the Euler identity implies

$$nf = \left(x_1 \frac{\partial f}{\partial x_1} + \dots + x_d \frac{\partial f}{\partial x_d} \right).$$

However $k = 1$ may not be sufficient for an inhomogeneous polynomial f , for example, take $f = x^2y^2 + x^5 + y^5 \in R$ where $R = \mathbb{C}\{x, y\}$. We claim that f does not belong to the ideal

$$\mathfrak{a} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2xy^2 + 5x^4, 2x^2y + 5y^4).$$

This is easily verified by comparing power series coefficients, or alternatively by working modulo the ideal $(x, y)^6$.

Briançon and Skoda answered Mather's question by proving that f^d belongs to the ideal generated by the partial derivatives, [SB]. Their proof uses the convergence of certain integrals. The absence of a purely algebraic proof for such an algebraic statement was highlighted by Hochster in his lectures on *Analytic methods in commutative algebra* and became, to quote Lipman and Teissier, "something of a scandal—perhaps even an insult—and certainly a challenge" for algebraists. The first algebraic proofs were found by Lipman and Teissier, and subsequently the result was extended to ideals in arbitrary regular local rings by Lipman and Sathaye:

Theorem 3.2 ([LiT, LiS]). *Let R be a regular ring, and \mathfrak{a} be an ideal generated by n elements. Then $\bar{\mathfrak{a}}^n \subseteq \mathfrak{a}$.*

Tight closure theory gives an extremely elementary proof for regular rings of positive characteristic:

Theorem 3.3 (Hochster-Huneke). *Let R be a Noetherian ring of prime characteristic and \mathfrak{a} be an ideal generated by n elements. Then*

$$\overline{\mathfrak{a}^n} \subseteq \mathfrak{a}^*.$$

In particular, if R is weakly F -regular, then $\overline{\mathfrak{a}^n} \subseteq \mathfrak{a}$.

Proof. It suffices to verify the assertion modulo each minimal prime of R , so we assume R is a domain. Let $\mathfrak{a} = (x_1, \dots, x_n)$ and $z \in \overline{\mathfrak{a}^n}$. By one of the characterizations of integral closure, there is a nonzero element $c \in R$ such that

$$cz^k \in \mathfrak{a}^{nk} \quad \text{for all } k \gg 0.$$

By the pigeonhole principle $\mathfrak{a}^{nk} \subseteq (x_1^k, \dots, x_n^k)$, so restricting k to $q = p^e$, we get $cz^q \in \mathfrak{a}^{[q]}$, and hence that $z \in \mathfrak{a}^*$. \square

The reader should have no difficulty in proving that for R and \mathfrak{a} as above,

$$\overline{\mathfrak{a}^{m+n}} \subseteq (\mathfrak{a}^{m+1})^* \quad \text{for all } m \geq 0.$$

There is a beautiful extension of the Briançon-Skoda theorem due to Aberbach and Huneke:

Theorem 3.4 ([AH3]). *Let R be an F -rational ring of positive characteristic, or a finitely generated algebra over a field of characteristic zero which has rational singularities. If \mathfrak{a} is an n -generated ideal of R , then*

$$\overline{\mathfrak{a}^{m+n}} \subseteq \mathfrak{a}^{m+1} \quad \text{for all } m \geq 0.$$

Lipman used the notion of adjoint ideals to obtain improved Briançon-Skoda theorems in [Li]. Improvements involving coefficient ideals may be found in [AH2], and for applications to Rees rings, see [AH1, AHT]. Rees and Sally studied the Briançon-Skoda theorem from another viewpoint in [RS], and Swanson's related work on joint reductions appears in [Sw1, Sw2]. In Wall's lectures on Mather's work, where it all began, it is amusing to find the sentence ([Wal, page 185]):

Once the seed of algebra is sown, it grows fast.

4. REDUCTION MODULO p

The first use of *reduction modulo p* methods we usually encounter is in the proof that cyclotomic polynomials are irreducible, (Dedekind, 1857) typically done in a graduate course in abstract algebra. The basic idea in Dedekind's proof, as in most reduction modulo p proofs, is to start with a statement in characteristic zero, reduce modulo p , and then exploit the Frobenius. The technique has proved extremely

useful in commutative algebra and yielded results for the equicharacteristic cases of the homological conjectures. We shall use reduction modulo p methods here to prove the Briançon-Skoda theorem for regular rings of characteristic zero, as well as the Hochster-Roberts theorem.

There are beautiful results relating the characteristic 0 and characteristic p properties of algebraic sets. Starting with a polynomial $f(x_1, \dots, x_d) \in \mathbb{Z}[x_1, \dots, x_d]$, the solution set of $f = 0$ in \mathbb{C}^d is a topological space. The Weil conjectures—now theorems of Grothendieck and Deligne—relate the Betti numbers of this topological space to the number of roots of $\bar{f} \in \mathbb{Z}/p[x_1, \dots, x_d]$ in the finite fields \mathbb{F}_{p^e} , where p^e is a prime power. Closer to the applications we have in mind here, is the following elementary result:

Proposition 4.1. *Consider a family of polynomials $f_1, \dots, f_n \in \mathbb{Z}[x_1, \dots, x_d]$. Then these polynomials have a common root over \mathbb{C} if and only if, for all but finitely many prime integers p , their images have a common root over the algebraic closure of \mathbb{Z}/p .*

Proof. If $(\alpha_1, \dots, \alpha_d) \in \mathbb{C}^d$ is a common root of the given polynomials, consider the subring $A = \mathbb{Z}[\alpha_1, \dots, \alpha_d]$ of \mathbb{C} . Let \mathfrak{m} be any maximal ideal of A . Then A/\mathfrak{m} is a field which is finitely generated as a \mathbb{Z} -algebra, and hence is a finite field, see [AM, Exercise 7.6]. Let p be the characteristic of A/\mathfrak{m} . Using $\bar{}$ to denote images modulo \mathfrak{m} , the point $(\bar{\alpha}_1, \dots, \bar{\alpha}_d) \in (A/\mathfrak{m})^d$ is a common root of the images of the f_i in $\mathbb{Z}/p[x_1, \dots, x_d]$. It remains to verify that A has maximal ideals containing infinitely many prime integers.

By the Noether's normalization lemma, $A_{\mathbb{Q}} = \mathbb{Q}[\alpha_1, \dots, \alpha_d]$ is an integral extension of a polynomial subring $\mathbb{Q}[y_1, \dots, y_t]$. Each α_i satisfies an equation of integral dependence over $\mathbb{Q}[y_1, \dots, y_t]$. Each of these d equations involves finitely many coefficients from \mathbb{Q} so, after localizing at a suitable integer r , we have an integral extension

$$\mathbb{Z}[y_1, \dots, y_t, 1/r] \subseteq A[1/r]. \quad (\#)$$

For every prime integer p not dividing r , there is a maximal ideal of $\mathbb{Z}[y_1, \dots, y_t, 1/r]$ which contains p . Since the extension (#) is integral, there exists a maximal ideal of $A[1/r]$ lying over every maximal ideal of $\mathbb{Z}[y_1, \dots, y_t, 1/r]$.

Conversely, if the polynomials do not have a common root in \mathbb{C}^d , then Hilbert's Nullstellensatz implies that f_1, \dots, f_n generate the unit ideal in $\mathbb{C}[x_1, \dots, x_d]$, i.e., that $\mathbb{C}[x_1, \dots, x_d]/(f_1, \dots, f_n) = 0$. But

$$\frac{\mathbb{C}[x_1, \dots, x_d]}{(f_1, \dots, f_n)} \cong \frac{\mathbb{Q}[x_1, \dots, x_d]}{(f_1, \dots, f_n)} \otimes_{\mathbb{Q}} \mathbb{C},$$

so $\mathbb{Q}[x_1, \dots, x_d]/(f_1, \dots, f_n) = 0$ since \mathbb{C} is faithfully flat over \mathbb{Q} . This implies that f_1, \dots, f_n generate the unit ideal in $\mathbb{Q}[x_1, \dots, x_d]$ as well and so, after multiplying

by a common denominator, we have an equation of the form

$$f_1g_1 + \cdots + f_n g_n = m$$

where $g_1, \dots, g_n \in \mathbb{Z}[x_1, \dots, x_d]$, and m is a nonzero integer. For every prime integer p not dividing m , the images of f_1, \dots, f_n generate the unit ideal in $\mathbb{Z}/p[x_1, \dots, x_d]$, and hence cannot have a common root over the algebraic closure of \mathbb{Z}/p . \square

One of the ingredients we will need in subsequent applications of reduction modulo p methods is the *generic freeness* lemma of Hochster-Roberts. A special case may be found in [Ma1, Chapter 8] and the general result is [HR, Lemma 8.1].

Lemma 4.2 (Generic freeness). *Let A be a Noetherian domain, B a finitely generated A -algebra, and C a finitely generated B -algebra. Let N be a finitely generated C -module, M a finitely generated B -submodule of N , and L a finitely generated A -submodule of N . Then there exists a nonzero element $a \in A$ such that the localization*

$$\left(\frac{N}{L+M} \right)_a$$

is a free A_a -module.

We next prove the Briançon-Skoda theorem for regular rings of characteristic zero.

Theorem 4.3. *Let \mathfrak{a} be an n -generated ideal of a regular ring which contains a field of characteristic zero. Then*

$$\overline{\mathfrak{a}^n} \subseteq \mathfrak{a}.$$

Proof. We first consider the case where the regular ring R is finitely generated over a field K , and return to the general case later in this section.

Consider R as a homomorphic image of a polynomial ring $T = K[x_1, \dots, x_d]$, say $R = T/(g_1, \dots, g_m)$. Let f_1, \dots, f_n be elements of T which map to generators of the ideal $\mathfrak{a} \subset R$. If the statement of the theorem is false, there exists $z \in T$ whose image in R belongs to the set $\overline{\mathfrak{a}^n} \setminus \mathfrak{a}$. Hence there exist elements $a_i \in (f_1, \dots, f_n)^{n_i}$ and $h_1, \dots, h_m \in (g_1, \dots, g_m)$ such that

$$z^l + a_1 z^{l-1} + \cdots + a_l = h_1 g_1 + \cdots + h_m g_m. \quad (\#)$$

Let A be a finitely generated \mathbb{Z} -subalgebra of K containing the coefficients of z, f_i, g_i, a_i , and h_i as polynomials in x_1, \dots, x_d , and also the coefficients of polynomials needed to express each a_i as an element of $(f_1, \dots, f_n)^{n_i}$.

In the next few steps, we shall replace A by its localization at finitely many elements. The conditions we require of A are preserved under further localization, and we do not change our notation for A . It is worth emphasizing that when we

adjoin the inverses of finitely many elements to the ring A , we retain the property that it is a finitely generated \mathbb{Z} -subalgebra of K . Consider the A -algebra

$$R_A = A[x_1, \dots, x_d]/(g_1, \dots, g_m).$$

Let $Q(A)$ denote the fraction field of A . The inclusion $Q(A) \rightarrow K$ is a flat homomorphism of A -modules so, upon tensoring with R_A , we get a flat homomorphism

$$R_A \otimes_A Q(A) \rightarrow R_A \otimes_A K \cong R.$$

The ring R is regular, so it follows that $R_A \otimes_A Q(A)$ is regular as well. Since $Q(A)$ is a field of characteristic zero, it follows that $R_A \otimes_A Q(A)$ is a smooth $Q(A)$ -algebra. After inverting an element of A , we may assume that $A \rightarrow R_A$ is smooth. Given a homomorphism from A to a field κ , we use the notation $R_\kappa = R_A \otimes_A \kappa$. For every such κ , the ring R_κ is smooth over κ by base change. In particular, R_κ is regular.

After localizing A at one element, we may assume that $R_A/(f_1, \dots, f_n)R_A$, and $R_A/(z, f_1, \dots, f_n)R_A$ are free A -modules. This ensures that $(f_1, \dots, f_n)R_A \otimes_A \kappa$ and $(z, f_1, \dots, f_n)R_A \otimes_A \kappa$ are ideals of the ring R_κ . Since the image of z in R does not belong to the ideal $(f_1, \dots, f_n)R$, after inverting one more element of A we may assume that

$$\frac{(z, f_1, \dots, f_n)R_A}{(f_1, \dots, f_n)R_A}$$

is a nonzero free A -module.

Let \mathfrak{m} be a maximal ideal of A . The field $\kappa = A/\mathfrak{m}$ is finitely generated as a \mathbb{Z} -algebra, hence is a finite field. We use z_κ to denote the image of z in R_κ . The freeness hypotheses give us the isomorphism

$$\frac{(z, f_1, \dots, f_n)R_A}{(f_1, \dots, f_n)R_A} \otimes_A \kappa \cong \frac{(z, f_1, \dots, f_n)R_\kappa}{(f_1, \dots, f_n)R_\kappa},$$

and also ensure that this module is nonzero, i.e., that $z_\kappa \notin (f_1, \dots, f_n)R_\kappa$. The image of equation (#) in R_κ implies that

$$z_\kappa \in \overline{(f_1, \dots, f_n)^n R_\kappa}.$$

But R_κ is a regular ring of positive characteristic, so this contradicts the characteristic p Briançon-Skoda theorem we proved earlier as Theorem 3.3. \square

For the general case, we will need a rather deep result on regular homomorphisms which we state next. See the appendix, particularly the subsection **Fibers and geometric regularity**, for relevant definitions.

Theorem 4.4 (General Néron Desingularization, [Po, Sw]). *Let $\varphi : R \rightarrow S$ be a regular homomorphism which factors through a finitely presented R -algebra R' .*

Then there exists an R' -algebra T such that the composite homomorphism $R \rightarrow T$ is smooth, and the following diagram commutes

$$\begin{array}{ccccc} R & \longrightarrow & R' & \longrightarrow & S \\ & & & \searrow & \uparrow \\ & & & & T \end{array} .$$

Consider the special case where R is the localization of a polynomial ring over a field and $\varphi : R \rightarrow \widehat{R}$ is the map to the completion. Since R is excellent, the map φ is regular, and so General Néron Desingularization applies here. This special case is a theorem of Artin and Rotthaus:

Theorem 4.5 ([AR]). *Let $R = K[x_1, \dots, x_d]_{\mathfrak{m}}$, i.e., R is the localization of a polynomial ring at its homogeneous maximal ideal \mathfrak{m} . Let \widehat{R} denote the \mathfrak{m} -adic completion of R . Then given any finitely generated R -subalgebra R' of \widehat{R} , the inclusion $R \rightarrow \widehat{R}$ factors as*

$$R \longrightarrow R' \longrightarrow T \longrightarrow \widehat{R}$$

where $R \rightarrow T$ is smooth.

Proof of Theorem 4.3 in the general case. Suppose the assertion of the theorem is false, there exists $z \in \overline{\mathfrak{a}^n}$ with $z \notin \mathfrak{a}$. Let \mathfrak{p} be a prime ideal containing the ideal $\mathfrak{a} : z$. Localizing at \mathfrak{p} and completing, we then have a counterexample in a power series ring $K[[x_1, \dots, x_d]]$ where K is a field of characteristic zero. Let $R = K[x_1, \dots, x_d]_{(x_1, \dots, x_d)}$. Let R' be an R -subalgebra of $\widehat{R} = K[[x_1, \dots, x_d]]$ which contains the element z , a set of generators for the ideal \mathfrak{a} , and the elements of \widehat{R} occurring in one equation demonstrating that $z \in \overline{\mathfrak{a}^n}$ in terms of the chosen generating set for \mathfrak{a} .

By the Artin-Rotthaus theorem, the maps $R \rightarrow \widehat{R}$ factors as

$$R \longrightarrow R' \longrightarrow T \longrightarrow \widehat{R}$$

where $R \rightarrow T$ is smooth. In particular T is regular, and the images of z and \mathfrak{a} in T also yield a counterexample, say $z_0 \in \overline{\mathfrak{a}_0^n} \setminus \mathfrak{a}_0$. Note that T is a regular ring of the form $T = S^{-1}B$, where B is a finitely generated algebra over the field K . Since $S^{-1}B$ is regular, S contains an element a of the defining ideal of the singular locus of B . The element z_0 , the generators for \mathfrak{a}_0 , and the elements occurring in one equation implying $z_0 \in \overline{\mathfrak{a}_0^n}$ involve finitely many elements, hence finitely many denominators from S , and we take b to be the product of these denominators. We then obtain a counterexample in the ring B_{ab} which is a regular ring finitely generated over the field K , but we have already proved the Briançon-Skoda theorem in this case. \square

It should be mentioned that this approach is not the only way to deduce characteristic zero results from positive characteristic theorems. Schoutens, [Sc], uses model-theoretic methods to obtain the Briançon-Skoda theorem for power series rings $\mathbb{C}[[X_1, \dots, X_d]]$ using the characteristic p version, Theorem 3.3.

We now return to the Hochster-Roberts theorem in characteristic zero. One subtle point is that the property that a ring S is a direct summand of R may not be preserved when we reduce modulo primes p , as we see in the following example.

Example 4.6. Consider the special case of Example 2.14 where $n = 2$ and $d = 3$, i.e, K is an infinite field, and $G = SL_2(K)$ acts on the polynomial ring $R = K[u, v, w, x, y, z]$, where $M \in G$ maps the entries of the matrix

$$X = \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix}$$

to those of MX . The ring of invariants for this action is $R^G = K[\Delta_1, \Delta_2, \Delta_3]$, where

$$\Delta_1 = vz - wy, \quad \Delta_2 = wx - uz, \quad \Delta_3 = uy - vx$$

are the size two minors of the matrix X . These minors are algebraically independent over K , and hence R^G is a polynomial ring. If K has characteristic zero, then G is linearly reductive, and hence R^G is a direct summand of R . We shall see that R^G is not a direct summand of R if K is any field of characteristic $p > 0$. Let $\mathfrak{a} = (\Delta_1, \Delta_2, \Delta_3)R$, and consider the local cohomology module

$$H_{\mathfrak{a}}^3(R) = \varinjlim_t \text{Ext}_R^3(R/\mathfrak{a}^t, R) \cong \varinjlim_{q=p^e} \text{Ext}_R^3(R/\mathfrak{a}^{[q]}, R),$$

where the isomorphism follows from the fact that the sequences of ideals $\{\mathfrak{a}_t\}_{t \in \mathbb{N}}$ and $\{\mathfrak{a}^{[p^e]}\}_{e \in \mathbb{N}}$ are cofinal. The projective resolution of R/\mathfrak{a} is

$$\mathbf{P}_{\bullet} = 0 \longrightarrow R^2 \xrightarrow{X} R^3 \xrightarrow{(\Delta_1 \ \Delta_2 \ \Delta_3)} R \longrightarrow 0.$$

Since the Frobenius $F : R \rightarrow R$ is flat, the complex $F^e(R) \otimes_R \mathbf{P}_{\bullet}$ is acyclic, and hence is a resolution of $R/\mathfrak{a}^{[q]}$. This implies that $\text{Ext}_R^3(R/\mathfrak{a}^{[q]}, R) = 0$ for all $q = p^e$, and hence that $H_{\mathfrak{a}}^3(R) = 0$. Using the description of $H_{\mathfrak{a}}^{\bullet}(R)$ as the cohomology of the Čech complex

$$0 \longrightarrow R \longrightarrow R_{\Delta_1} \oplus R_{\Delta_2} \oplus R_{\Delta_3} \longrightarrow R_{\Delta_2\Delta_3} \oplus R_{\Delta_3\Delta_1} \oplus R_{\Delta_1\Delta_2} \longrightarrow R_{\Delta_1\Delta_2\Delta_3} \longrightarrow 0,$$

$H_{\mathfrak{a}}^3(R) = 0$ implies that

$$\frac{1}{\Delta_1\Delta_2\Delta_3} = \frac{r_1}{(\Delta_2\Delta_3)^t} + \frac{r_2}{(\Delta_3\Delta_1)^t} + \frac{r_3}{(\Delta_1\Delta_2)^t}$$

for $r_i \in R$ and $t \in \mathbb{N}$. Clearing denominators gives us

$$(\Delta_1\Delta_2\Delta_3)^{t-1} = r_1\Delta_1^t + r_2\Delta_2^t + r_3\Delta_3^t \in (\Delta_1^t, \Delta_2^t, \Delta_3^t)R.$$

Since $(\Delta_1 \Delta_2 \Delta_3)^{t-1} \notin (\Delta_1^t, \Delta_2^t, \Delta_3^t)R^G$, it follows that R^G is not a direct summand of R .

We now return to the Hochster-Roberts theorem in the form:

Theorem 4.7. *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K , and let S be an \mathbb{N} -graded subring of R which is a direct summand of R . Then S is Cohen-Macaulay.*

Proof of Theorem 2.7. Let y_1, \dots, y_d be a homogeneous system of parameters for the ring S . Suppose there exist $s_i \in S$ with

$$s_1 y_1 + \dots + s_m y_m = 0 \quad (\#)$$

and $s_m \notin (y_1, \dots, y_{m-1})S$ then, since S is a direct summand of R , we also have $s_m \notin (y_1, \dots, y_{m-1})R$.

The ring S is module-finite over its polynomial subring $B = K[y_1, \dots, y_d]$, say $S = \sum_{i=1}^N B u_i$ where u_i are finitely many homogeneous elements of S . For all $1 \leq i, j, t \leq N$, there exist elements $c_{ijt} \in B$ such that $u_i u_j = \sum_{t=1}^N c_{ijt} u_t$. Let A be the ring obtained by adjoining to \mathbb{Z} the following elements of K :

- the coefficients that occur when each y_i and each u_i is written as a polynomial in x_1, \dots, x_n ,
- coefficients occurring when each c_{ijt} is written as a polynomial in y_1, \dots, y_d ,
- for each s_i , the coefficients of the polynomials in y_1, \dots, y_d needed in one equation expressing s_i as a B -linear combination of u_1, \dots, u_N .

Note that A is a finitely generated \mathbb{Z} -algebra of K . Set

$$B_A = A[y_1, \dots, y_d], \quad S_A = \sum_{i=1}^N B_A u_i, \quad \text{and} \quad R_A = A[x_1, \dots, x_n].$$

Then S_A is a subring of R_A and is module-finite over its subring B_A . Since $s_m \notin (y_1, \dots, y_{m-1})R$, after replacing A by a localization at one element, we may assume that

$$\frac{(s_m, y_1, \dots, y_{m-1})R_A}{(y_1, \dots, y_{m-1})R_A}$$

is a nonzero free A -module. After localizing A at an element, we may also assume that each of

$$\frac{R_A}{(s_m, y_1, \dots, y_{m-1})R_A}, \quad \frac{R_A}{(y_1, \dots, y_{m-1})R_A}, \quad S_A/B_A, \quad \text{and} \quad R_A/S_A$$

is a free A -module.

Let \mathfrak{m} be a maximal ideal of A . Then $\kappa = A/\mathfrak{m}$ is a finitely generated \mathbb{Z} -algebra, hence a finite field. We use the notation M_κ to denote $M \otimes_A \kappa$, and $\bar{}$ to denote the image of an element modulo \mathfrak{m} . Note that $R_\kappa = \kappa[\bar{x}_1, \dots, \bar{x}_n]$ and

$B_\kappa = \kappa[\bar{y}_1, \dots, \bar{y}_d]$ are polynomial rings over the field κ , and that the freeness hypotheses ensure that $B_\kappa \subseteq S_\kappa \subseteq R_\kappa$. The image of equation (#) in S_κ gives us

$$\bar{s}_1 \bar{y}_1 + \dots + \bar{s}_m \bar{y}_m = 0,$$

which is a relation on the homogeneous system of parameters $\bar{y}_1, \dots, \bar{y}_d$ of S_κ . By Theorem 2.9 it follows that

$$\bar{s}_m \in ((\bar{y}_1, \dots, \bar{y}_{m-1}) S_\kappa)^*.$$

Since $S_\kappa \subseteq R_\kappa$ and R_κ is a polynomial ring, we have

$$\bar{s}_m \in ((\bar{y}_1, \dots, \bar{y}_{m-1}) R_\kappa)^* = (\bar{y}_1, \dots, \bar{y}_{m-1}) R_\kappa,$$

but this contradicts the condition that

$$\frac{(s_m, y_1, \dots, y_{m-1}) R_A}{(y_1, \dots, y_{m-1}) R_A} \otimes_A \kappa \cong \frac{(\bar{s}_m, \bar{y}_1, \dots, \bar{y}_{m-1}) R_\kappa}{(\bar{y}_1, \dots, \bar{y}_{m-1}) R_\kappa}$$

is a nonzero module. □

5. APPENDIX

5.1. The structure theory of complete local rings. Every ring R admits a ring homomorphism $\varphi : \mathbb{Z} \rightarrow R$ where $\varphi(1)$ is the unit element of R . The kernel of this homomorphism is an ideal of \mathbb{Z} generated by a unique nonnegative integer $\text{char}(R)$, the *characteristic* of R . A local ring (R, \mathfrak{m}, K) is *equicharacteristic* or of *equal characteristic* if $\text{char}(R) = \text{char}(K)$.

For a local ring (R, \mathfrak{m}, K) the possible values of $\text{char}(K)$ and $\text{char}(R)$ are:

- $\text{char}(K) = \text{char}(R) = 0$, in which case $\mathbb{Q} \subseteq R$.
- $\text{char}(K) = \text{char}(R) = p > 0$, in which case $\mathbb{Z}/p\mathbb{Z} \subseteq R$.
- $\text{char}(K) = p > 0$ and $\text{char}(R) = 0$. As an example of this, take R to be the ring of p -adic integers.
- $\text{char}(K) = p > 0$ and $\text{char}(R) = p^e$ for some $e \geq 2$. In this case R is not a reduced ring.

If (R, \mathfrak{m}) is a local ring containing a field, we say that a field $L \subseteq R$ is a *coefficient field* for R if the composition

$$L \hookrightarrow R \twoheadrightarrow R/\mathfrak{m}$$

is an isomorphism. Proofs of the following theorems due to Cohen maybe found in [Ma1, Chapter 11] and [Ma1, Appendix].

Theorem 5.1. *Let (R, \mathfrak{m}, K) be a complete local ring containing a field. Then:*

- (1) *The ring R contains a coefficient field.*
- (2) *If $L \subseteq R$ is a coefficient field, then R is a homomorphic image of a formal power series ring over L , i.e., $R \cong L[[T_1, \dots, T_n]]/\mathfrak{a}$.*

- (3) If $L \subseteq R$ is a coefficient field and x_1, \dots, x_d is a system of parameters for R , then the subring $A = L[[x_1, \dots, x_d]]$ of R is isomorphic to a formal power series ring $L[[T_1, \dots, T_d]]$, and R is a finitely generated A -module.
- (4) The ring R is regular if and only if it is isomorphic to a formal power series ring $L[[T_1, \dots, T_d]]$ over some coefficient field L .

In the case where (R, \mathfrak{m}, K) does not contain a field, for the sake of simplicity we make the assumption that R is an integral domain. This ensures that the only possibility is $\text{char}(K) = p > 0$ and $\text{char}(R) = 0$, and this case is usually referred to as *mixed characteristic*. The role of a coefficient field is now replaced by that of a discrete valuation ring (V, pV) , which serves as a *coefficient ring*.

A regular local ring (R, \mathfrak{m}, K) of mixed characteristic is *unramified* if $p \notin \mathfrak{m}^2$, and is *ramified* if $p \in \mathfrak{m}^2$.

Theorem 5.2. *Let (R, \mathfrak{m}, K) be a complete local domain with $\text{char}(R) = 0$ and $\text{char}(K) = p > 0$. Then there exists a discrete valuation ring (V, pV) which is a subring of R , such $V \subseteq R$ induces an isomorphism $V/pV \rightarrow R/\mathfrak{m}$.*

- (1) The ring R is a homomorphic image of a formal power series ring over V , i.e., $R \cong V[[T_1, \dots, T_n]]/\mathfrak{a}$.
- (2) If p, x_2, \dots, x_d is a system of parameters for the ring R , consider the subring $A = V[[x_2, \dots, x_d]]$. Then A is isomorphic to a formal power series ring $V[[T_2, \dots, T_d]]$, and R is a finitely generated A -module.
- (3) The ring R is an unramified regular local ring if and only if it is isomorphic to a formal power series ring $V[[T_2, \dots, T_d]]$.
- (4) The ring R is a ramified regular local ring if and only if it is isomorphic to $V[[T_1, \dots, T_d]]/(p - f)$ where f is an element in the square of the maximal ideal of $V[[T_1, \dots, T_d]]$.

5.2. Excellent rings. The class of *excellent rings* was introduced by Grothendieck to circumvent some pathological behavior that can occur in the larger class of Noetherian rings. The precise definition is somewhat technical in nature, but is satisfied by most Noetherian rings that are likely to be encountered by mathematicians working in algebraic geometry, number theory, or several variable complex analysis. For a detailed treatment and proofs of results summarized here, the reader should consult [EGA], particularly §7, and [Ma1, Chapter 13]. The expository article [Ma2] provides a nice introduction to the theory of excellent rings of characteristic zero. Various examples of non-excellent rings were constructed by Nagata, and are included as an appendix in his book [Na2].

Definition 5.3. A ring R is said to be *excellent* if

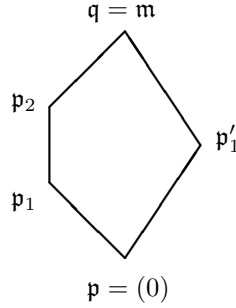
- (1) R is Noetherian,

- (2) R is universally catenary,
- (3) for every prime ideal $\mathfrak{p} \in \text{Spec } R$, the formal fibers of the local ring $R_{\mathfrak{p}}$ are geometrically regular, and
- (4) for every finitely generated R -algebra S , the regular locus of the ring S , i.e., the set $\{\mathfrak{p} \in \text{Spec } S : S_{\mathfrak{p}} \text{ is a regular local ring}\}$, is an open subset of $\text{Spec } S$.

Of course, we need to define some of the terms occurring above.

Universally catenary rings. A ring R is *catenary* if for all prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ of R , all saturated chains of prime ideals joining \mathfrak{p} and \mathfrak{q} have the same length. A ring R is *universally catenary* if every finitely generated algebra over R is a catenary ring.

Several attempts had been made to prove that all Noetherian rings were catenary, until Nagata constructed the first examples of noncatenary Noetherian rings in 1956. He constructed a local integral domain (R, \mathfrak{m}) of dimension 3 which is not catenary; R has saturated chains of prime ideals joining $\mathfrak{p} = (0)$ and $\mathfrak{q} = \mathfrak{m}$ of lengths 2 and 3, as illustrated in the diagram below:



Theorem 5.4 (Dimension formula). *Let $R \subseteq S$ be integral domains such that R is universally catenary and S is a finitely generated R -algebra. Let $\mathfrak{q} \in \text{Spec } S$ and $\mathfrak{p} = \mathfrak{q} \cap R \in \text{Spec } R$. Then*

$$\text{height } \mathfrak{q} + \text{tr.deg }_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}) = \text{height } \mathfrak{p} + \text{tr.deg }_R S,$$

where $\kappa(\mathfrak{p})$ denotes the field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ and $\kappa(\mathfrak{q}) = S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}}$, and by $\text{tr.deg }_R S$ we mean the transcendence degree of fraction field of S over the fraction field of R .

Ratliff showed that the above dimension formula, in a sense, characterizes universally catenary rings, see [Ra1, Ra2].

Fibers and geometric regularity. For a ring homomorphism $\varphi : R \rightarrow S$, by the *fiber* of φ at a prime $\mathfrak{p} \in \text{Spec } R$, we mean the ring $S \otimes_R \kappa(\mathfrak{p})$ where $\kappa(\mathfrak{p})$ denotes the field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Note that the inverse image of \mathfrak{p} under the induced

map $\text{Spec } S \longrightarrow \text{Spec } R$ is homeomorphic to $\text{Spec } S \otimes_R \kappa(\mathfrak{p})$, which explains the use (rather, the misuse) of the word “fiber.”

For a local ring (R, \mathfrak{m}) , the *formal fibers* of R are the fibers of the homomorphism $R \longrightarrow \widehat{R}$, where \widehat{R} denotes the completion of R at its maximal ideal \mathfrak{m} . If \mathfrak{a} is an ideal of R , then $\widehat{R/\mathfrak{a}} = \widehat{R}/\mathfrak{a}\widehat{R}$, and so a formal fiber of the ring R/\mathfrak{a} is also a formal fiber of the ring R .

A K -algebra R is *geometrically regular* if $R \otimes_K L$ is a regular ring for every finite extension L of the field K . This is equivalent to the condition that $R \otimes_K L$ is regular for every finite purely inseparable extension L of K .

A ring homomorphism $\varphi : R \longrightarrow S$ is *regular* if it is flat and for all $\mathfrak{p} \in \text{Spec } R$, the fiber $\kappa(\mathfrak{p}) \otimes_R S$ is geometrically regular.

A ring homomorphism $\varphi : R \longrightarrow S$ is *smooth* if it is regular, and S is finitely presented over the image of R . In this case, for every R -algebra T , the homomorphism $R \otimes_R T \longrightarrow S \otimes_R T$ is also smooth.

The ubiquity of excellent rings. The result below explains why the Noetherian rings we encounter are almost always excellent rings:

Theorem 5.5. *If a ring R is obtained by adjoining finitely many variables to a field or a complete discrete valuation ring, taking a homomorphic image, and localizing at some multiplicative set, then R is an excellent ring. More precisely,*

- (1) *Every complete local ring (in particular, every field) is excellent. The ring of convergent power series over \mathbb{R} or \mathbb{C} is excellent. A Dedekind domain whose field of fractions has characteristic zero (e.g. \mathbb{Z}) is excellent.*
- (2) *A finitely generated algebra over an excellent ring is excellent; in particular, a homomorphic image of an excellent ring is excellent.*
- (3) *A localization of an excellent ring is excellent.*

There is an interesting class of rings of characteristic $p > 0$ which are excellent by the following theorem of Kunz:

Theorem 5.6. [Ku1, Ku2] *For a ring R of characteristic $p > 0$, let $F : R \longrightarrow R$ be the Frobenius homomorphism. If R is module-finite over $F(R)$, then R is excellent.*

Some properties of excellent rings.

Theorem 5.7. *If R is an excellent ring, then the normal locus as well as the Cohen-Macaulay locus, i.e.,*

$$\{\mathfrak{p} \in \text{Spec } R : R_{\mathfrak{p}} \text{ is normal}\} \quad \text{and} \quad \{\mathfrak{p} \in \text{Spec } R : R_{\mathfrak{p}} \text{ is Cohen-Macaulay}\},$$

are open subsets of $\text{Spec } R$.

Nagata defined a ring R to be a *pseudo-geometric ring* if for every prime $\mathfrak{p} \in \text{Spec } R$, and for every finite extension field K of the field of fractions of R/\mathfrak{p} , the integral closure of R/\mathfrak{p} in K is a finitely generated R/\mathfrak{p} -module. Examples of Noetherian rings which did not satisfy this property were first constructed by Akizuki in [Ak]. In honor of the Japanese school of commutative algebra, Grothendieck renamed pseudo-geometric rings as *anneaux universellement japonais*, or universally Japanese rings [EGA, §7.7]. At some point they were again renamed, and are now called *Nagata rings*.

Theorem 5.8. *An excellent ring is a Nagata ring.*

The excellence property also ensures that certain properties of a local ring R are inherited by its \mathfrak{m} -adic completion \widehat{R} , and this is the essence of the next theorem:

Theorem 5.9. *Let R be an excellent local ring with maximal ideal \mathfrak{m} .*

- (1) *If R is reduced (i.e., has no nonzero nilpotent elements) then \widehat{R} , its \mathfrak{m} -adic completion, is also a reduced ring.*
- (2) *If R is a domain then, by (1) above, \widehat{R} is a reduced ring. In this case, there is a bijection between the minimal primes of \widehat{R} and maximal ideals of R' , where R' denotes the integral closure of R in its field of fractions. In particular, \widehat{R} is a domain if and only if R' is local.*
- (3) *If R is a normal ring, then \widehat{R} is also a normal ring.*

We conclude with the following theorem of Hochster:

Theorem 5.10. [Ho2] *Let S be a reduced excellent local ring with maximal ideal \mathfrak{m} . Then for all $n > 0$, there is an \mathfrak{m} -primary ideal $\mathfrak{a}_n \subseteq \mathfrak{m}^n$ such that the ring S/\mathfrak{a}_n is injective as a module over itself (i.e., is a zero-dimensional Gorenstein ring).*

As a consequence of this, if R is a module-finite extension ring of S and $\mathfrak{a}R \cap S = \mathfrak{a}$ for all ideals \mathfrak{a} of S , then S is a direct summand of R as an S -module.

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