

A CONNECTEDNESS RESULT IN POSITIVE CHARACTERISTIC

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Dedicated to Professor Paul Roberts on the occasion of his sixtieth birthday

ABSTRACT. Let (R, \mathfrak{m}) be a complete local ring of dimension at least two, which contains a separably closed coefficient field of positive characteristic. Using a vanishing theorem of Peskine-Szpiro, Lyubeznik proved that the local cohomology module $H_{\mathfrak{m}}^1(R)$ is Frobenius-torsion if and only if the punctured spectrum of R is connected in the Zariski topology. We give a simple proof of this theorem and, more generally, a formula for the number of connected components in terms of the Frobenius action on $H_{\mathfrak{m}}^1(R)$.

1. INTRODUCTION

All rings considered in this note are commutative and Noetherian. We give a simple proof of the following result due to Lyubeznik:

Theorem 1.1. [Ly2, Corollary 4.6] *Let (R, \mathfrak{m}) be a complete local ring of dimension at least two, with a separably closed coefficient field of positive characteristic. Then the e -th iteration of the Frobenius map*

$$F: H_{\mathfrak{m}}^1(R) \longrightarrow H_{\mathfrak{m}}^1(R)$$

is zero for $e \gg 0$ if and only if $\text{Spec } R \setminus \{\mathfrak{m}\}$ is connected in the Zariski topology.

We also obtain, by similar methods, the following theorem:

Theorem 1.2. *Let (R, \mathfrak{m}) be a complete local ring of positive dimension, with an algebraically closed coefficient field of positive characteristic. Then the number of connected components of $\text{Spec } R \setminus \{\mathfrak{m}\}$ is*

$$1 + \dim_K \bigcap_{e \in \mathbb{N}} F^e(H_{\mathfrak{m}}^1(R)).$$

In Section 5 we describe how this provides an algorithm to determine the number of geometrically connected components of projective algebraic sets defined over a finite field: computer algebra algorithms for primary decomposition can be used to determine the number of connected components over finite extensions of the fields \mathbb{F}_p or \mathbb{Q} , but not over the algebraic closures of these fields. In the case of characteristic zero, de Rham cohomology allows for the computation of the number of geometrically connected components via D -module methods, [Wal], and we show that the Frobenius provides analogous methods in the case of positive characteristic.

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Theorem 1.1 is obtained in [Ly2] as a corollary of the following two theorems of Lyubeznik and Peskine-Szpiro:

Theorem 1.3. [Ly2, Theorem 1.1] *Let (A, \mathfrak{M}) be a regular local ring containing a field of positive characteristic, and let \mathfrak{A} be an ideal of A . Then $H_{\mathfrak{A}}^i(A) = 0$ if and only if there exists an integer $e \geq 1$ such that the e -th Frobenius iteration*

$$F^e : H_{\mathfrak{M}}^{\dim A - i}(A/\mathfrak{A}) \longrightarrow H_{\mathfrak{M}}^{\dim A - i}(A/\mathfrak{A})$$

is the zero map.

Theorem 1.4. [PS, Chapter III, Theorem 5.5] *Let (A, \mathfrak{M}) be a complete regular local ring with a separably closed coefficient field of positive characteristic, and let \mathfrak{A} be an ideal of A . Then $H_{\mathfrak{A}}^i(A) = 0$ for $i \geq \dim A - 1$ if and only if $\dim(A/\mathfrak{A}) \geq 2$ and $\text{Spec}(A/\mathfrak{A}) \setminus \{\mathfrak{M}\}$ is connected.*

Our proof of Theorem 1.1 is “simple” in the sense that it does not rely on vanishing theorems such as those of [PS]—indeed, the only ingredient, aside from elementary considerations, is the local duality theorem. Results analogous to Theorem 1.4 were proved by Hartshorne in the projective case [HaR, Theorem 7.5], and by Ogus in equicharacteristic zero using de Rham cohomology [Og, Corollary 2.11]. Combining these results, one has:

Theorem 1.5. *Let (A, \mathfrak{M}) be a regular local ring containing a field, and let \mathfrak{A} be an ideal of A . Then $H_{\mathfrak{A}}^i(A) = 0$ for $i \geq \dim A - 1$ if and only if*

- (1) $\dim(A/\mathfrak{A}) \geq 2$, and
- (2) $\text{Spec}(A/\mathfrak{A}) \setminus \{\mathfrak{M}\}$ is formally geometrically connected (see Definition 2.1).

Huneke and Lyubeznik [HL, Theorem 2.9] gave a characteristic free proof of this using a generalization of a result of Faltings, [Fa, Satz 1]. Some other applications of local cohomology theory which yield strong results on the connectedness properties of algebraic varieties may be found in the papers [BR] and [HH], where the authors obtain generalizations of Faltings’ connectedness theorem.

For the convenience of the reader, we include an Appendix with some facts about Frobenius actions; see Section 6.

2. PRELIMINARY REMARKS

Notation: When R is the homomorphic image of a ring A , we use upper-case letters $\mathfrak{P}, \mathfrak{Q}, \mathfrak{M}, \mathfrak{A}, \mathfrak{B}$ for ideals of A , and corresponding lower-case letters $\mathfrak{p}, \mathfrak{q}, \mathfrak{m}, \mathfrak{a}, \mathfrak{b}$ for their images in R .

Definition 2.1. Let (R, \mathfrak{m}) be a local ring. A field $K \subseteq R$ is a *coefficient field* for R if the composition $K \hookrightarrow R \twoheadrightarrow R/\mathfrak{m}$ is an isomorphism. Every complete local ring containing a field has a coefficient field.

We recall some notions from [Ra, Chapitre VIII]. Let (R, \mathfrak{m}, K) be a local ring and let $\overline{f(T)} \in K[T]$ denote the image of a polynomial $f(T) \in R[T]$. Then R is *Henselian* if for every monic polynomial $f(T) \in R[T]$, every factorization of $\overline{f(T)}$ as a product of relatively prime monic polynomials in $K[T]$ lifts to a factorization of $f(T)$ as a product of monic polynomials in $R[T]$. Hensel’s Lemma is precisely the statement that every complete local ring is Henselian. The *Henselization* of a local ring R is a local ring R^{h} , with the property that every local homomorphism from R to a Henselian local ring factors uniquely through R^{h} . The ring R^{h} is obtained by

taking the direct limit of all local étale extensions S of R for which $(R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ induces an isomorphism of residue fields $R/\mathfrak{m} \xrightarrow{\cong} S/\mathfrak{n}$.

A local ring (R, \mathfrak{m}, K) is said to be *strictly Henselian* if it is Henselian and its residue field K is separably closed. It is easily seen that R is strictly Henselian if and only if every monic polynomial $f(T) \in R[T]$ for which $\overline{f(T)} \in K[T]$ is separable splits into linear factors in $R[T]$. Every local ring has a *strict Henselization* R^{sh} , such that every local homomorphism from R to a strictly Henselian ring factors through R^{sh} . The strict Henselization of a field K is its separable closure K^{sep} . In general, the strict Henselization of a local ring (R, \mathfrak{m}, K) is obtained by fixing an embedding $\iota: K \longrightarrow K^{\text{sep}}$, and taking the direct limit of local étale extensions (S, \mathfrak{n}, L) of (R, \mathfrak{m}, K) with $L \hookrightarrow K^{\text{sep}}$, for which the induced map $K \longrightarrow L \longrightarrow K^{\text{sep}}$ agrees with $\iota: K \longrightarrow K^{\text{sep}}$.

The *punctured spectrum* of a local ring (R, \mathfrak{m}) is the set $\text{Spec } R \setminus \{\mathfrak{m}\}$, with the topology induced by the Zariski topology on $\text{Spec } R$. We say that *the punctured spectrum of R is formally geometrically connected* if the punctured spectrum of \widehat{R}^{sh} , the completion of the strict Henselization of the completion of R , is connected. If R is an \mathbb{N} -graded ring which is finitely generated over a field $R_0 = K$, then $\text{Proj } R$ is said to be *geometrically connected* if $\text{Proj}(R \otimes_K K^{\text{sep}})$ is connected.

Definition 2.2. Let \mathfrak{a} be an ideal of a ring R . A ring homomorphism $\varphi: R \longrightarrow S$ induces a map of local cohomology modules $H_{\mathfrak{a}}^i(R) \xrightarrow{\varphi} H_{\mathfrak{a}S}^i(S)$. In particular, if R contains a field of characteristic $p > 0$, then the Frobenius homomorphism $F: R \longrightarrow R$ induces an additive map

$$H_{\mathfrak{a}}^i(R) \xrightarrow{F} H_{\mathfrak{a}^{[p]}}^i(R) = H_{\mathfrak{a}}^i(R),$$

called the *Frobenius action* on $H_{\mathfrak{a}}^i(R)$. An element $\eta \in H_{\mathfrak{a}}^i(R)$ is *F-torsion* if $F^e(\eta) = 0$ for some $e \in \mathbb{N}$. The module $H_{\mathfrak{a}}^i(R)$ is *F-torsion* if each element is *F-torsion*. The image of F^e need not be an R -module, but it is a K -vector space when K is perfect. In this case the *F-stable* part of $H_{\mathfrak{a}}^i(R)$ is the vector space

$$H_{\mathfrak{a}}^i(R)_{\text{st}} = \bigcap_{e \in \mathbb{N}} F^e(H_{\mathfrak{a}}^i(R)).$$

Some results about *F-torsion* modules and *F-stable* subspaces are summarized in Section 6. For a very general theory of *F-modules*, we refer the reader to [Ly1].

Remark 2.3. Consider a local ring (R, \mathfrak{m}) of positive dimension. The punctured spectrum of R is disconnected if and only if the minimal primes of R can be partitioned into two sets $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ and $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ such that $\text{rad}(\mathfrak{p}_i + \mathfrak{q}_j) = \mathfrak{m}$ for all pairs $\mathfrak{p}_i, \mathfrak{q}_j$. Consider the graph Γ whose vertices are the minimal primes of R , and there is an edge between minimal primes \mathfrak{p} and \mathfrak{p}' if and only if $\text{rad}(\mathfrak{p} + \mathfrak{p}') \neq \mathfrak{m}$. It follows that the punctured spectrum of R is connected if and only if the graph Γ is connected. If the graph Γ is connected, take a spanning tree, i.e., a connected acyclic subgraph, containing all the vertices of Γ . This spanning tree must contain a vertex \mathfrak{p}_i with only one edge, so $\Gamma \setminus \{\mathfrak{p}_i\}$ is connected as well.

Let $\mathfrak{P}_1, \dots, \mathfrak{P}_n$ be incomparable prime ideals of a local domain A . Then their images $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are precisely the minimal primes of the ring $R = A/(\mathfrak{P}_1 \cap \dots \cap \mathfrak{P}_n)$. From the above discussion, we conclude that if the punctured spectrum of R is connected, then there exists i such that the punctured spectrum of the ring

$$A/(\mathfrak{P}_1 \cap \dots \cap \widehat{\mathfrak{P}_i} \cap \dots \cap \mathfrak{P}_n)$$

is connected as well.

Theorems 1.1 and 1.2 assert that connectedness issues for $\text{Spec } R \setminus \{\mathfrak{m}\}$ are determined by the Frobenius action on $H_{\mathfrak{m}}^1(R)$. We next record an observation about the length of $H_{\mathfrak{m}}^1(R)$.

Proposition 2.4. *Let (R, \mathfrak{m}) be a local ring which is a homomorphic image of a Gorenstein domain. Then $H_{\mathfrak{m}}^1(R)$ has finite length if and only if $\text{ann}_R \mathfrak{p} = 0$ for every prime ideal \mathfrak{p} of R with $\dim R/\mathfrak{p} = 1$.*

Proof. If $\dim R = 0$, then $H_{\mathfrak{m}}^1(R) = 0$, and R has no primes with $\dim R/\mathfrak{p} = 1$. If $\dim R = 1$, then $H_{\mathfrak{m}}^1(R)$ has infinite length and $\dim R/\mathfrak{p} = 1$ for some minimal prime \mathfrak{p} of R . For the rest of the proof we hence assume that $\dim R \geq 2$.

Let $R = A/\Omega$ where A is a Gorenstein domain. Localizing A at the inverse image of \mathfrak{m} , we may assume that (A, \mathfrak{M}) is a local ring. Using local duality over A , the module $H_{\mathfrak{m}}^1(R) = H_{\mathfrak{M}}^1(A/\Omega)$ has finite length if and only if $\text{Ext}_A^{\dim A - 1}(A/\Omega, A)$ has finite length as an A -module. Since $\text{Ext}_A^{\dim A - 1}(A/\Omega, A)$ is finitely generated, this is equivalent to the vanishing of

$$\text{Ext}_A^{\dim A - 1}(A/\Omega, A)_{\mathfrak{P}} = \text{Ext}_{A_{\mathfrak{P}}}^{\dim A - 1}(A_{\mathfrak{P}}/\Omega A_{\mathfrak{P}}, A_{\mathfrak{P}})$$

for all $\mathfrak{P} \in \text{Spec } A \setminus \{\mathfrak{M}\}$. Using local duality over the Gorenstein local ring $(A_{\mathfrak{P}}, \mathfrak{P}A_{\mathfrak{P}})$, this is equivalent to the vanishing of

$$H_{\mathfrak{P}A_{\mathfrak{P}}}^{\dim A_{\mathfrak{P}} - \dim A + 1}(A_{\mathfrak{P}}/\Omega A_{\mathfrak{P}}) = H_{\mathfrak{p}R_{\mathfrak{p}}}^{\dim A_{\mathfrak{P}} - \dim A + 1}(R_{\mathfrak{p}})$$

for all $\mathfrak{P} \in \text{Spec } A \setminus \{\mathfrak{M}\}$. This local cohomology module vanishes for $\mathfrak{P} \notin V(\Omega)$. Since $\dim A_{\mathfrak{P}} - \dim A + 1 \leq 0$ for $\mathfrak{P} \in \text{Spec } A \setminus \{\mathfrak{M}\}$, we need only consider primes $\mathfrak{P} \in V(\Omega)$ with $\dim A_{\mathfrak{P}} = \dim A - 1$. Since A is a catenary local domain, $\dim A_{\mathfrak{P}}$ equals $\dim A - 1$ precisely when $\dim A/\mathfrak{P} = 1$, equivalently $\dim R/\mathfrak{p} = 1$. Hence $H_{\mathfrak{m}}^1(R)$ has finite length if and only if $H_{\mathfrak{p}R_{\mathfrak{p}}}^0(R_{\mathfrak{p}}) = H_{\mathfrak{p}}^0(R)$ vanishes for all $\mathfrak{p} \in \text{Spec } R$ with $\dim R/\mathfrak{p} = 1$, i.e., if and only if $\text{ann}_R \mathfrak{p} = 0$ for all \mathfrak{p} with $\dim R/\mathfrak{p} = 1$. \square

3. MAIN RESULTS

Theorem 3.1. *Let (R, \mathfrak{m}) be a strictly Henselian local domain containing a field of positive characteristic. If R is a homomorphic image of a Gorenstein domain and $\dim R \geq 2$, then $H_{\mathfrak{m}}^1(R)$ is F -torsion.*

Proof. Suppose there exists $\eta \in H_{\mathfrak{m}}^1(R)$ which is not F -torsion. Since R is a domain, Proposition 2.4 implies that $H_{\mathfrak{m}}^1(R)$ has finite length. Hence for all integers $e \geq 0$, the element $F^e(\eta)$ belongs to the R -module spanned by $\eta, F(\eta), F^2(\eta), \dots, F^{e-1}(\eta)$. Amongst all equations of the form

$$(3.1.1) \quad F^{e+k}(\eta) + r_1 F^{e+k-1}(\eta) + \dots + r_e F^k(\eta) = 0$$

with $r_i \in R$ for all i , choose one where the number of nonzero coefficients r_i that occur is minimal. We claim that r_e must be a unit. Note that $H_{\mathfrak{m}}^1(R)$ is killed by $\mathfrak{m}^{q'}$ for some $q' = p^{e'}$. If $r_e \in \mathfrak{m}$, then applying $F^{e'}$ to equation (3.1.1), we get

$$F^{e'+e+k}(\eta) + r_1^{q'} F^{e'+e+k-1}(\eta) + \dots + r_e^{q'} F^{e'+k}(\eta) = 0.$$

But $r_e^{q'} F^{e'+k}(\eta) \in \mathfrak{m}^{q'} H_{\mathfrak{m}}^1(R) = 0$, so this is an equation with fewer nonzero coefficients, contradicting the minimality assumption. This shows that $r_e \in R$ is a unit.

Since η is not F -torsion, neither is $F^k(\eta)$, so after replacing η if necessary, we have an equation of the form

$$(3.1.2) \quad F^e(\eta) + r_1 F^{e-1}(\eta) + \cdots + r_e \eta = 0$$

where r_e is a unit and $\eta \in H_{\mathfrak{m}}^1(R)$ is not F -torsion. Let $\eta = [(y_1/x_1, \dots, y_d/x_d)]$ where $H_{\mathfrak{m}}^1(R)$ is regarded as the cohomology of a Čech complex on a system of parameters x_1, \dots, x_d for R . Then (3.1.2) implies that there exists $r_{e+1} \in R$ such that each $y_i/x_i \in R_{x_i}$ is a root of the polynomial

$$f(T) = T^{p^e} + r_1 T^{p^{e-1}} + \cdots + r_e T + r_{e+1} \in R[T].$$

Now $f'(T) = r_e$ is a unit, so $\overline{f(T)} \in R/\mathfrak{m}[T]$ is a separable polynomial. Since R is strictly Henselian, the polynomial $f(T)$ splits in $R[T]$, and hence any root of $f(T)$ in the fraction field of R is an element of R . In particular, $y_1/x_1 = \cdots = y_d/x_d$ is an element of R , and so $\eta = 0$. \square

We next prove the connectedness criterion, Theorem 1.1. By Proposition 6.1, the module $H_{\mathfrak{m}}^1(R)$ is F -torsion if and only if there exists e such that $F^e(H_{\mathfrak{m}}^1(R)) = 0$. In view of this, the following theorem is equivalent to Theorem 1.1.

Theorem 3.2. *Let (R, \mathfrak{m}) be a local ring with $\dim R > 0$, which contains a field of positive characteristic. Then $H_{\mathfrak{m}}^1(R)$ is F -torsion if and only if $\dim R \geq 2$ and the punctured spectrum of R is formally geometrically connected.*

Proof. Quite generally, for a local ring (R, \mathfrak{m}) we have $H_{\mathfrak{m}}^i(\widehat{R}) = H_{\mathfrak{m}}^i(R)$. Moreover, $S = \widehat{R}^{\text{sh}}$ is a faithfully flat extension of R , and $H_{\mathfrak{m}}^i(R) \otimes_R S \cong H_{\mathfrak{m}_S}^i(S)$ is F -torsion if and only if $H_{\mathfrak{m}}^i(R)$ is F -torsion. Hence we may assume that R is a complete local ring with a separably closed coefficient field.

Suppose that $H_{\mathfrak{m}}^1(R)$ is F -torsion. The local cohomology module $H_{\mathfrak{m}}^{\dim R}(R)$ is not F -torsion by Proposition 6.2, so $\dim R \geq 2$. Let \mathfrak{a} and \mathfrak{b} be ideals of R such that $\mathfrak{a} + \mathfrak{b}$ is \mathfrak{m} -primary and $\mathfrak{a} \cap \mathfrak{b} = 0$. Let

$$x_1 = y_1 + z_1, \quad \dots, \quad x_d = y_d + z_d$$

be a system of parameters for R where $y_i \in \mathfrak{a}$ and $z_i \in \mathfrak{b}$. Since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b} = 0$, we have $y_i z_j = 0$ for all i, j , and hence

$$y_i(y_j + z_j) = y_j(y_i + z_i).$$

These relations give an element of $H_{\mathfrak{m}}^1(R)$ regarded as the cohomology of a Čech complex on x_1, \dots, x_d , namely

$$\eta = \left[\left(\frac{y_1}{x_1}, \dots, \frac{y_d}{x_d} \right) \right] \in H_{\mathfrak{m}}^1(R).$$

The hypotheses imply that $F^e(\eta) = 0$ for some e , so there exists $q = p^e$ and $r \in R$ such that $(y_i/x_i)^q = r$ in R_{x_i} for all $1 \leq i \leq d$. Hence there exists $t \in \mathbb{N}$ such that $x_i^t y_i^q = r x_i^{q+t}$, i.e.,

$$(y_i + z_i)^t y_i^q = r (y_i + z_i)^{q+t}.$$

But $y_i z_i = 0$, so these equations simplify to give $(1-r)y_i^{q+t} = r z_i^{q+t}$. Since R is a local ring, either r or $1-r$ must be a unit. If r is a unit, then $z_i^{q+t} \in \mathfrak{a}$ for all i , and so \mathfrak{a} is \mathfrak{m} -primary. Similarly if $1-r$ is a unit, then \mathfrak{b} is \mathfrak{m} -primary. This proves that the punctured spectrum of R is connected.

For the converse, assume that $\dim R \geq 2$ and that the punctured spectrum of R is connected. Let \mathfrak{n} denote the nilradical of R . Note that $\text{Spec } R$ is homeomorphic to $\text{Spec } R/\mathfrak{n}$. Moreover, \mathfrak{n} supports a Frobenius action and is F -torsion. The long exact sequence of local cohomology relating $H_{\mathfrak{m}}^1(R)$ and $H_{\mathfrak{m}}^1(R/\mathfrak{n})$ implies that if $H_{\mathfrak{m}}^1(R/\mathfrak{n})$ is F -torsion, then so is $H_{\mathfrak{m}}^1(R)$, and hence there is no loss of generality in assuming that R is reduced. Let $R = A/(\mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n)$ where $\mathfrak{P}_1, \dots, \mathfrak{P}_n$ are incomparable prime ideals of a power series ring $A = K[[x_1, \dots, x_m]]$ over a separably closed field K . We use induction on n to prove that $H_{\mathfrak{m}}^1(R)$ is F -torsion; the case $n = 1$ follows from Theorem 3.1, so we assume $n > 1$ below.

If $\dim R/\mathfrak{p}_i = 1$ for some i , then $\text{Spec } R \setminus \{\mathfrak{m}\}$ is the disjoint union of $V(\mathfrak{p}_i) \setminus \{\mathfrak{m}\}$ and $V(\mathfrak{p}_1 \cap \cdots \cap \widehat{\mathfrak{p}_i} \cap \cdots \cap \mathfrak{p}_n) \setminus \{\mathfrak{m}\}$, contradicting the connectedness assumption. Hence $\dim R/\mathfrak{p}_i \geq 2$ for all i . By Remark 2.3, after relabeling the minimal primes if necessary, we may assume that the punctured spectrum of A/\mathfrak{Q} is connected where $\mathfrak{Q} = \mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_n$. The short exact sequence

$$0 \longrightarrow A/(\mathfrak{P}_1 \cap \mathfrak{Q}) \longrightarrow A/\mathfrak{P}_1 \oplus A/\mathfrak{Q} \longrightarrow A/(\mathfrak{P}_1 + \mathfrak{Q}) \longrightarrow 0$$

induces a long exact sequence of local cohomology modules containing the piece

$$(3.2.1) \quad H_{\mathfrak{M}}^0(A/(\mathfrak{P}_1 + \mathfrak{Q})) \longrightarrow H_{\mathfrak{M}}^1(A/(\mathfrak{P}_1 \cap \mathfrak{Q})) \longrightarrow H_{\mathfrak{M}}^1(A/\mathfrak{P}_1) \oplus H_{\mathfrak{M}}^1(A/\mathfrak{Q}).$$

Since $\text{rad}(\mathfrak{P}_1 + \mathfrak{P}_i) \neq \mathfrak{M}$ for some $i > 1$, it follows that $\dim A/(\mathfrak{P}_1 + \mathfrak{Q}) \geq 1$. Proposition 6.2 now implies that $H_{\mathfrak{M}}^0(A/(\mathfrak{P}_1 + \mathfrak{Q}))$ is F -torsion. By the inductive hypothesis, $H_{\mathfrak{M}}^1(A/\mathfrak{P}_1)$ and $H_{\mathfrak{M}}^1(A/\mathfrak{Q})$ are F -torsion as well. The exact sequence (3.2.1) implies that $H_{\mathfrak{M}}^1(A/(\mathfrak{P}_1 \cap \mathfrak{Q})) = H_{\mathfrak{m}}^1(R)$ is F -torsion. \square

The following lemma will be used in the proof of Theorem 1.2.

Lemma 3.3. *Let (R, \mathfrak{m}) be a complete local domain with an algebraically closed coefficient field of positive characteristic. Then $H_{\mathfrak{m}}^1(R)_{\text{st}}$, the F -stable part of the module $H_{\mathfrak{m}}^1(R)$, is zero.*

Proof. If $\dim R = 0$, then $H_{\mathfrak{m}}^1(R) = 0$, and if $\dim R \geq 2$, then the assertion follows from Theorem 3.1. The remaining case is $\dim R = 1$. Theorem 6.3 implies that $H_{\mathfrak{m}}^1(R)_{\text{st}}$ has a vector space basis η_1, \dots, η_r such that $F(\eta_i) = \eta_i$.

Let $\eta \in H_{\mathfrak{m}}^1(R)_{\text{st}}$ be an element with $F(\eta) = \eta$. Considering $H_{\mathfrak{m}}^1(R)$ as the cohomology of a suitable Čech complex, let η be the class of y/x in $R_x/R = H_{\mathfrak{m}}^1(R)$, where $y \in R$ and $x \in \mathfrak{m}$. Since $F(\eta) = \eta$, there exists $r \in R$ such that

$$\left(\frac{y}{x}\right)^p - \frac{y}{x} - r = 0,$$

and so $y/x \in R_x$ is a root of the polynomial $f(T) = T^p - T - r \in R[T]$. The polynomial $f(T) \in K[T]$ is separable and R is strictly Henselian, so $f(T)$ splits in $R[T]$. Since y/x is a root of $f(T)$ in the fraction field of R , it must then be an element of R , and hence $\eta = 0$. \square

Proof of Theorem 1.2. We may assume R to be reduced by Proposition 6.5. First consider the case where the punctured spectrum of R is connected. If $\dim R \geq 2$, then $H_{\mathfrak{m}}^1(R)$ is F -torsion by Theorem 3.2, so $H_{\mathfrak{m}}^1(R)_{\text{st}} = 0$. If $\dim R = 1$, then R is a domain, and Lemma 3.3 implies that $H_{\mathfrak{m}}^1(R)_{\text{st}} = 0$.

We continue by induction on the number of connected components of the punctured spectrum of R . If the punctured spectrum of R is disconnected, then $R = A/(\mathfrak{A} \cap \mathfrak{B})$ where (A, \mathfrak{M}) is a power series ring over the field K , and \mathfrak{A} and \mathfrak{B} are

radical ideals of A which are not \mathfrak{M} -primary, but $\mathfrak{A} + \mathfrak{B}$ is \mathfrak{M} -primary. There is a short exact sequence

$$0 \longrightarrow A/(\mathfrak{A} \cap \mathfrak{B}) \longrightarrow A/\mathfrak{A} \oplus A/\mathfrak{B} \longrightarrow A/(\mathfrak{A} + \mathfrak{B}) \longrightarrow 0.$$

Since $H_{\mathfrak{M}}^0(A/\mathfrak{A}) = H_{\mathfrak{M}}^0(A/\mathfrak{B}) = H_{\mathfrak{M}}^0(A/(\mathfrak{A} + \mathfrak{B})) = 0$, the resulting exact sequence of local cohomology gives us

$$0 \longrightarrow H_{\mathfrak{M}}^0(A/(\mathfrak{A} + \mathfrak{B})) \longrightarrow H_{\mathfrak{M}}^1(A/(\mathfrak{A} \cap \mathfrak{B})) \longrightarrow H_{\mathfrak{M}}^1(A/\mathfrak{A}) \oplus H_{\mathfrak{M}}^1(A/\mathfrak{B}) \longrightarrow 0.$$

By Theorem 6.4, we have a K -vector space isomorphism

$$H_{\mathfrak{m}}^1(R)_{\text{st}} = H_{\mathfrak{M}}^1(A/(\mathfrak{A} \cap \mathfrak{B}))_{\text{st}} \cong H_{\mathfrak{M}}^0(A/(\mathfrak{A} + \mathfrak{B}))_{\text{st}} \oplus H_{\mathfrak{M}}^1(A/\mathfrak{A})_{\text{st}} \oplus H_{\mathfrak{M}}^1(A/\mathfrak{B})_{\text{st}}.$$

Since $H_{\mathfrak{M}}^0(A/(\mathfrak{A} + \mathfrak{B}))_{\text{st}} = K$ by Proposition 6.2, the inductive hypothesis completes the proof. \square

We next record the graded versions of the results proved in this section:

Theorem 3.4. *Let R be an \mathbb{N} -graded ring of positive dimension, which is finitely generated over a field $R_0 = K$ of characteristic $p > 0$.*

- (1) *If R is a domain with $\dim R \geq 2$, and K is separably closed, then $H_{\mathfrak{m}}^1(R)$ is F -torsion.*
- (2) *The module $H_{\mathfrak{m}}^1(R)$ is F -torsion if and only if $\dim R \geq 2$ and $\text{Proj } R$ is geometrically connected.*
- (3) *Let K be a perfect field, and let \overline{K} denote its algebraic closure. Then the number of connected components of $\text{Proj}(R \otimes_K \overline{K})$ is*

$$1 + \dim_K H_{\mathfrak{m}}^1(R)_{\text{st}} = 1 + \dim_K ([H_{\mathfrak{m}}^1(R)]_0)_{\text{st}}.$$

Proof. (1) Note that $H_{\mathfrak{m}}^1(R)$ is a \mathbb{Z} -graded R -module, and that

$$F: [H_{\mathfrak{m}}^1(R)]_n \longrightarrow [H_{\mathfrak{m}}^1(R)]_{np} \quad \text{for all } n \in \mathbb{Z}.$$

The module $H_{\mathfrak{m}}^1(R)$ has finite length, so all elements of $H_{\mathfrak{m}}^1(R)$ of positive or negative degree are F -torsion; it remains to show that elements $\eta \in [H_{\mathfrak{m}}^1(R)]_0$ are F -torsion as well. Let η be a element of $[H_{\mathfrak{m}}^1(R)]_0$ which is not F -torsion. As in the proof of Theorem 3.1, after a change of notation we may assume that

$$F^e(\eta) + r_1 F^{e-1}(\eta) + \cdots + r_e \eta = 0$$

where all r_i are in $[R]_0 = K$, and r_e is nonzero. Let $\eta = [(y_1/x_1, \dots, y_d/x_d)]$ where $H_{\mathfrak{m}}^1(R)$ is regarded as the cohomology of a homogeneous Čech complex. Then there exists $r_{e+1} \in K$ such that $y_i/x_i \in R_{x_i}$ is a root of the polynomial

$$f(T) = T^{p^e} + r_1 T^{p^{e-1}} + \cdots + r_e T + r_{e+1} \in K[T].$$

But $f(T)$ is a separable polynomial, so it splits in $K[T]$. The element $y_i/x_i = y_j/x_j$ is a root of $f(T)$ in the fraction field of R , so it must be one of the roots of $f(T)$ in K . It follows that $\eta = 0$, which completes the proof of (1).

The proof of (2) is now similar to that of Theorem 3.2 and is left to the reader. For (3), note that $F^e(H_{\mathfrak{m}}^1(R))$ is a K -vector space since K is perfect, and that

$$\dim_K H_{\mathfrak{m}}^1(R)_{\text{st}} = \dim_{\overline{K}} H_{\mathfrak{m}}^1(R \otimes_K \overline{K})_{\text{st}}.$$

Thus we may assume $K = \overline{K}$, and the proof is similar to that of Theorem 1.2. \square

Remark 3.5. Theorem 3.4.3 generalizes, in the case of positive characteristic, the well-known fact that the number of connected components of $X = \text{Proj } R$ is

$$\dim_K H^0(X, \mathcal{O}_X) = 1 + \dim_K [H_{\mathfrak{m}}^1(R)]_0,$$

where R is an \mathbb{N} -graded *reduced* ring of positive dimension, which is finitely generated over an algebraically closed field $R_0 = K$. The point is that in this case the Frobenius is bijective on $[H_{\mathfrak{m}}^1(R)]_0$. To see this, let

$$\eta = \left[\left(\frac{y_1}{x_1}, \dots, \frac{y_d}{x_d} \right) \right] \in [H_{\mathfrak{m}}^1(R)]_0$$

be an element with $F(\eta) = 0$, where $H_{\mathfrak{m}}^1(R)$ is computed as the cohomology of a suitable Čech complex. Then there exists a homogeneous element $r \in R$ with $(y_i/x_i)^p = r$ in R_{x_i} for all $1 \leq i \leq d$. Such an element r must have degree zero, and hence must be an element of K . But then $r^{1/p} \in K$, and, since R is reduced, $y_i/x_i = r^{1/p}$ for all i . It follows that

$$\eta = [(r^{1/p}, \dots, r^{1/p})] = 0.$$

To complete the argument, note that $[H_{\mathfrak{m}}^1(R)]_0$ is a finite dimensional K -vector space, and that if $\eta_1, \dots, \eta_n \in [H_{\mathfrak{m}}^1(R)]_0$ are linearly independent, then so are $F(\eta_1), \dots, F(\eta_n)$. It follows that $F: [H_{\mathfrak{m}}^1(R)]_0 \rightarrow [H_{\mathfrak{m}}^1(R)]_0$ is surjective.

4. F -PURITY

A ring homomorphism $\varphi: R \rightarrow S$ is *pure* if $\varphi \otimes 1: R \otimes_R M \rightarrow S \otimes_R M$ is injective for every R -module M . If R is a ring containing a field of characteristic $p > 0$, then R is *F -pure* if the Frobenius homomorphism $F: R \rightarrow R$ is pure. The notion was introduced by Hochster and Roberts in the course of their study of rings of invariants in [HR1, HR2].

Examples of F -pure rings include regular rings of positive characteristic and their pure subrings. If \mathfrak{a} is generated by square-free monomials in the variables x_1, \dots, x_n and K is a field of positive characteristic, then $K[x_1, \dots, x_n]/\mathfrak{a}$ is F -pure.

Goto and Watanabe [GW] classified one-dimensional F -pure rings: let (R, \mathfrak{m}) be a local ring of positive characteristic such that $R/\mathfrak{m} = K$ is algebraically closed, $F: R \rightarrow R$ is finite, and $\dim R = 1$. Then R is F -pure if and only if

$$\widehat{R} \cong K[[x_1, \dots, x_n]]/(x_i x_j \mid i < j).$$

Two-dimensional F -pure rings have attracted a lot of attention: Watanabe [Wat1] proved that F -pure normal Gorenstein local rings of dimension two are either rational double points, simple elliptic singularities, or cusp singularities. He also classified two-dimensional normal \mathbb{N} -graded rings R over an algebraically closed field R_0 , in terms of \mathbb{Q} -divisors on the curve $\text{Proj } R$, [Wat2]. In [MS] Mehta and Srinivas obtained a classification of two-dimensional F -pure normal singularities in terms of the resolution of the singularity. Hara completed the classification of two-dimensional normal F -pure singularities in terms of the dual graph of the minimal resolution of the singularity, [HaN].

The results of Section 3 imply that over separably closed fields, F -pure domains of dimension two are Cohen-Macaulay. The point is that if R is an F -pure ring, then the Frobenius action $F: H_{\mathfrak{m}}^i(R) \rightarrow H_{\mathfrak{m}}^i(R)$ is an injective map.

Corollary 4.1. *Let R be a local ring with $\dim R \geq 2$, which contains a field of positive characteristic. If R is F -pure and the punctured spectrum of R is formally geometrically connected, then $\text{depth } R \geq 2$.*

In particular, if R is a complete local F -pure domain of dimension two, with a separably closed coefficient field, then R is Cohen-Macaulay.

Proof. An F -pure ring is reduced, so $H_{\mathfrak{m}}^0(R) = 0$. By Theorem 3.1, $H_{\mathfrak{m}}^1(R)$ is F -torsion. Since R is F -pure, it follows that $H_{\mathfrak{m}}^1(R) = 0$. \square

In the graded case, we similarly have:

Corollary 4.2. *Let R be an \mathbb{N} -graded ring with $\dim R \geq 2$, which is finitely generated over a field R_0 of positive characteristic. If R is F -pure and $\text{Proj } R$ is geometrically connected, then $\text{depth } R \geq 2$.*

The ring R below is a graded F -pure domain of dimension two, and depth one. The issue is that $\text{Proj } R$ is connected though not geometrically connected.

Example 4.3. Let K be a field of characteristic $p > 2$, and $a \in K$ an element such that $\sqrt{a} \notin K$. Let $R = K[x, y, x\sqrt{a}, y\sqrt{a}]$. The domain R has a presentation

$$R = K[x, y, u, v]/(u^2 - ax^2, v^2 - ay^2, uv - axy, vx - uy),$$

and if K^{sep} denotes the separable closure of K , then

$$R \otimes_K K^{\text{sep}} \cong K^{\text{sep}}[x, y, u, v]/(u - x\sqrt{a}, v - y\sqrt{a})(u + x\sqrt{a}, v + y\sqrt{a}).$$

Using a change of variables, $R \otimes_K K^{\text{sep}} \cong K^{\text{sep}}[x', y', u', v']/(x', y')(u', v')$. Since $(x', y')(u', v')$ is a square-free monomial ideal, $R \otimes_K K^{\text{sep}}$ is F -pure and it follows that R is F -pure. However, R is not Cohen-Macaulay since x, y is a homogeneous system of parameters with a non-trivial relation

$$(x\sqrt{a})y = (y\sqrt{a})x.$$

Using the Čech complex on x, y to compute $H_{\mathfrak{m}}^1(R)$, we see that it is a 1-dimensional K -vector space generated by the element

$$\eta = \left[\begin{pmatrix} x\sqrt{a} & y\sqrt{a} \\ x & y \end{pmatrix} \right] \in H_{\mathfrak{m}}^1(R)$$

corresponding to the relation above. Given $e \in \mathbb{N}$, let $p^e = 2k + 1$. Then

$$F^e(\eta) = a^k \eta,$$

which is a nonzero element of $H_{\mathfrak{m}}^1(R)$. Consequently $H_{\mathfrak{m}}^1(R)$ is not F -torsion, corresponding to the fact that $\text{Proj } R$ is not geometrically connected.

The corollaries obtained in this section imply that over a separably closed field, a graded or complete local F -pure domain of dimension two is Cohen-Macaulay. We record an example which shows that this is not true for rings of higher dimension.

Example 4.4. Let K be a field of characteristic $p > 0$, and take

$$A = K[x_1, \dots, x_d]/(x_1^d + \dots + x_d^d)$$

where $d \geq 3$. Let R be the Segre product of A and the polynomial ring $B = K[s, t]$. Then $\dim R = d$, and the Künneth formula for local cohomology implies that

$$H_{\mathfrak{m}_R}^{d-1}(R) \cong [H_{\mathfrak{m}_A}^{d-1}(A)]_0 \otimes_K [B]_0 \cong K,$$

so R is not Cohen-Macaulay. If $p \equiv 1 \pmod{d}$, then A is F -pure by [HR2, Proposition 5.21]; hence $A \otimes_K B$ and its direct summand R are F -pure as well.

5. ALGORITHMIC ASPECTS

Let R be an \mathbb{N} -graded ring, which is finitely generated over a finite field $R_0 = K$. We wish to determine the number of geometrically connected components of the scheme $\text{Proj } R$, i.e., the number of connected components of $\text{Proj}(R \otimes_K \overline{K})$, or, equivalently, of $\text{Proj}(R \otimes_K K^{\text{sep}})$. While primary decomposition algorithms such as those of [EHV], [GTZ], or [SY], may be used to determine the connected components of $\text{Proj } R$, there is computationally no hope of “determining” the connected components over the algebraic closure, \overline{K} . However, simply finding their number is much easier: by Theorem 3.4, this is $1 + \dim_K([H_m^1(R)]_0)_{\text{st}}$. Computing this number involves three steps.

- (1) Finding a good presentation of $[H_m^1(R)]_0$;
- (2) Determining the Frobenius action on $[H_m^1(R)]_0$ in terms of this presentation;
- (3) Computing the dimension of the F -stable part, $([H_m^1(R)]_0)_{\text{st}}$.

If $R = A/\mathfrak{A}$ for a polynomial ring A , we first replace \mathfrak{A} by an ideal that has the same radical as \mathfrak{A} , but does not have the homogeneous maximal ideal \mathfrak{M} as an associated prime. This can be done by saturating \mathfrak{A} with respect to \mathfrak{M} ; if desired, one may simply compute the radical of \mathfrak{A} , but this is often computationally expensive. Now, since \mathfrak{M} is not associated to \mathfrak{A} , one can find a homogeneous system of parameters x_1, \dots, x_d for R such that each x_i is a nonzerodivisor on R .

The length ℓ of $[H_m^1(R)]_0$ may be computed by computing the length of its graded dual $[\text{Ext}_A^{n-1}(R, A(-n))]_0$, where $\dim A = n$. Of course, if this length is zero, then $X_{\overline{K}}$ is connected. Consider the Koszul cohomology modules

$$H^1(x_1^t, \dots, x_d^t; R) = \frac{\{(a_1, \dots, a_d) \in R^d \mid a_i x_j^t = a_j x_i^t \text{ for all } i < j\}}{\{(rx_1^t, \dots, rx_d^t) \mid r \in R\}}.$$

These modules have an \mathbb{N} -grading, where for homogeneous elements $a_i \in R$, we define the degree of $[(a_1, \dots, a_d)] \in H^1(x_1^t, \dots, x_d^t; R)$ as

$$\deg[(a_1, \dots, a_d)] = \deg a_i - \deg x_i^t,$$

which is independent of i . This ensures that for each t , the map

$$\begin{aligned} H^1(x_1^t, \dots, x_d^t; R) &\longrightarrow H^1(x_1^{t+1}, \dots, x_d^{t+1}; R) \\ [(a_1, \dots, a_d)] &\longmapsto [(a_1 x_1, \dots, a_d x_t)] \end{aligned}$$

preserves degrees. The module $H_m^1(R)$ is the direct limit of these Koszul cohomology modules, and the assumption that the x_i are nonzerodivisors ensures that the maps in the direct limit system are injective. The modules $H^1(x_1^t, \dots, x_d^t; R)$ may be computed for increasing values of t , until we arrive at an integer N such that

$$\ell([H^1(x_1^N, \dots, x_d^N; R)]_0) = \ell.$$

This gives us a presentation for $[H_m^1(R)]_0 = [H^1(x_1^N, \dots, x_d^N; R)]_0$, in terms of which we now analyze the Frobenius map. Replacing the x_i by their powers if needed, assume that $N = 1$. Let

$$\alpha = [(a_1, \dots, a_d)] \in [H^1(x_1, \dots, x_d; R)]_0,$$

in which case, $F(\alpha) = [(a_1^p, \dots, a_d^p)] \in [H^1(x_1^p, \dots, x_d^p; R)]_0$. Since the map

$$[H^1(x_1, \dots, x_d; R)]_0 \longrightarrow [H^1(x_1^p, \dots, x_d^p; R)]_0$$

coming from the direct limit system is bijective, it follows that $a_i^p \in x_i^{p-1}R$ for each $1 \leq i \leq d$. Setting $b_i = a_i^p/x_i^{p-1}$, we arrive at

$$F(\alpha) = [(b_1, \dots, b_d)] \in [H^1(x_1, \dots, x_d; R)]_0.$$

Using this description of Frobenius action on the finite dimensional K -vector space $[H_m^1(R)]_0 = [H^1(x_1, \dots, x_d; R)]_0$, it is now straightforward to compute the ranks of the vector spaces

$$[H_m^1(R)]_0 \supseteq F([H_m^1(R)]_0) \supseteq F^2([H_m^1(R)]_0) \supseteq \dots,$$

and hence of the F -stable part, $([H_m^1(R)]_0)_{\text{st}}$.

6. APPENDIX: F -TORSION MODULES AND F -STABLE VECTOR SPACES

Let R be a commutative ring containing a field K of characteristic $p > 0$. A *Frobenius action* on an R -module M is an additive map $F: M \rightarrow M$ such that $F(rm) = r^p F(m)$ for all $r \in R$ and $m \in M$. In this case, $\ker F$ is a submodule of M , and we have an ascending sequence of submodules of M ,

$$\ker F \subseteq \ker F^2 \subseteq \ker F^3 \subseteq \dots$$

The union of these is the F -nilpotent submodule of M , denoted $M_{\text{nil}} = \bigcup_{e \in \mathbb{N}} \ker F^e$. We say M is F -torsion if $M_{\text{nil}} = M$.

Proposition 6.1. *Let (R, \mathfrak{m}) be a local ring containing a field of positive characteristic, and let M be an Artinian R -module with a Frobenius action. Then there exists $e \in \mathbb{N}$ such that $F^e(M_{\text{nil}}) = 0$.*

Hence an Artinian module M is F -torsion if and only if $F^e(M) = 0$ for some e .

Proof. This is proved in [HS, Proposition 1.11] under the hypothesis that R is a complete local ring with a perfect coefficient field. The general case may be concluded from this, but a more elegant approach is via Lyubeznik's theory of F -modules; see [Ly1, Proposition 4.4]. \square

If R is a ring containing a perfect field K of positive characteristic and M is an R -module with a Frobenius action, then $F(M)$ is a K -vector space, and we have a descending sequence of K -vector spaces

$$F(M) \supseteq F^2(M) \supseteq F^3(M) \supseteq \dots$$

The F -stable part of M is the vector space $M_{\text{st}} = \bigcap_{e \in \mathbb{N}} F^e(M)$.

Proposition 6.2. *Let (R, \mathfrak{m}, K) be a local ring of dimension d which contains a field of positive characteristic.*

- (1) $H_{\mathfrak{m}}^0(R)$ is F -torsion if and only if $d > 0$.
- (2) $H_{\mathfrak{m}}^d(R)$ is not F -torsion.
- (3) If $d = 0$ and K is perfect, then $H_{\mathfrak{m}}^0(R)_{\text{st}} = R_{\text{st}} = K$.

Proof. (1) If $d = 0$, then $H_{\mathfrak{m}}^0(R) = R$, which is not F -torsion. If $d > 0$, then $H_{\mathfrak{m}}^0(R)$ is contained in \mathfrak{m} . Since every element of $H_{\mathfrak{m}}^0(R)$ is killed by a power of \mathfrak{m} , it follows that each element is nilpotent. (See also [Ly2, Corollary 4.6(a)].)

(2) View $H_{\mathfrak{m}}^d(R)$ as the cohomology of a Čech complex on a system of parameters x_1, \dots, x_d for R , and let $\eta = [1 + (x_1, \dots, x_d)] \in H_{\mathfrak{m}}^d(R)$. For all $e_0 \in \mathbb{N}$, the collection of elements $F^e(\eta)$ with $e > e_0$ generates $H_{\mathfrak{m}}^d(R)$ as an R -module. Hence $F^{e_0}(\eta)$ cannot be zero by Grothendieck's nonvanishing theorem.

(3) Since \mathfrak{m} is nilpotent in this case, for integers $e \gg 0$ we have

$$F^e(H_{\mathfrak{m}}^0(R)) = F^e(R) = \{x^{p^e} \mid x \in R\} = \{(y+z)^{p^e} \mid y \in K, z \in \mathfrak{m}\} = K. \quad \square$$

Theorem 6.3. *Let (R, \mathfrak{m}) be a local ring with a perfect coefficient field K of positive characteristic. Let M be an Artinian R -module with a Frobenius action. Then M_{st} is a finite dimensional K -vector space, and $F: M_{\text{st}} \rightarrow M_{\text{st}}$ is an automorphism of the Abelian group M_{st} .*

If K is algebraically closed, then there exists a K -basis e_1, \dots, e_n for M_{st} such that $F(e_i) = e_i$ for all $1 \leq i \leq n$.

Proof. For the finiteness assertion, see [HS, Theorem 1.12] or [Ly1, Proposition 4.9]. It is easily seen that $F: M_{\text{st}} \rightarrow M_{\text{st}}$ is an automorphism whenever M_{st} is finite dimensional. The existence of the special basis when K is algebraically closed follows from [Di, Proposition 5, page 233]. \square

Theorem 6.4. [HS, Theorem 1.13] *Let (R, \mathfrak{m}) be a complete local ring with an algebraically closed coefficient field of positive characteristic. Let L, M, N be R -modules with Frobenius actions such that we have a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \longrightarrow & 0 \\ & & F \downarrow & & F \downarrow & & F \downarrow & & \\ 0 & \longrightarrow & L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \longrightarrow & 0 \end{array}$$

with exact rows. If L is Noetherian and N is Artinian, then the F -stable parts form a short exact sequence

$$0 \longrightarrow L_{\text{st}} \longrightarrow M_{\text{st}} \longrightarrow N_{\text{st}} \longrightarrow 0.$$

Proposition 6.5. *Let (R, \mathfrak{m}, K) be a complete local ring with an algebraically closed coefficient field of positive characteristic. Let \mathfrak{n} denote the nilradical of R . Then for all $i \geq 0$, the natural map $H_{\mathfrak{m}}^i(R) \rightarrow H_{\mathfrak{m}}^i(R/\mathfrak{n})$, when restricted to F -stable subspaces, gives an isomorphism*

$$H_{\mathfrak{m}}^i(R)_{\text{st}} \xrightarrow{\cong} H_{\mathfrak{m}}^i(R/\mathfrak{n})_{\text{st}}.$$

Proof. Let k be an integer such that $\mathfrak{n}^{p^k} = 0$. The short exact sequence

$$0 \longrightarrow \mathfrak{n} \longrightarrow R \longrightarrow R/\mathfrak{n} \longrightarrow 0$$

induces a long exact sequence of local cohomology modules

$$\dots \longrightarrow H_{\mathfrak{m}}^i(\mathfrak{n}) \xrightarrow{\alpha} H_{\mathfrak{m}}^i(R) \xrightarrow{\beta} H_{\mathfrak{m}}^i(R/\mathfrak{n}) \xrightarrow{\gamma} H_{\mathfrak{m}}^{i+1}(\mathfrak{n}) \longrightarrow \dots$$

Consider an element $\mu \in \ker(\beta) \cap H_{\mathfrak{m}}^i(R)_{\text{st}}$. Then $\mu \in \text{image}(\alpha)$, so $F^k(\mu) = 0$. The Frobenius action on $H_{\mathfrak{m}}^i(R)_{\text{st}}$ is an automorphism, so $\mu = 0$, and hence the map $H_{\mathfrak{m}}^i(R)_{\text{st}} \rightarrow H_{\mathfrak{m}}^i(R/\mathfrak{n})_{\text{st}}$ is injective.

To complete the proof it suffices, by Theorem 6.3, to consider an element $\eta \in H_{\mathfrak{m}}^i(R/\mathfrak{n})_{\text{st}}$ with $F(\eta) = \eta$, and prove that it lies in the image of $H_{\mathfrak{m}}^i(R)_{\text{st}}$. Now $\gamma(\eta) \in H_{\mathfrak{m}}^{i+1}(\mathfrak{n})$ so $F^k(\gamma(\eta)) = 0$, and therefore $F^k(\eta) = \eta \in \ker(\gamma)$.

Let $\eta = \beta(\mu)$ for some element $\mu \in H_{\mathfrak{m}}^i(R)$. Then $\beta(F(\mu) - \mu) = 0$, which implies that $F(\mu) - \mu \in \text{image}(\alpha)$. Consequently $F^k(F(\mu) - \mu) = 0$, which shows that $F^{k+1}(\mu) = F^k(\mu)$, and hence that $F^k(\mu) \in H_{\mathfrak{m}}^i(R)_{\text{st}}$. Since

$$\beta(F^k(\mu)) = F^k(\beta(\mu)) = F^k(\eta) = \eta,$$

we are done. \square

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