Math 6320, Assignment 3

Due in class: Tuesday, March 6

1. Determine the Galois group of $x^6 + 3 \in \mathbb{Q}[x]$.

2. Let $G$ be the Galois group of $x^5 - 7 \in \mathbb{Q}[x]$. Is $G$ solvable? Is it abelian?

3. Determine all fields that lie between $\mathbb{Q}$ and $\mathbb{Q}(e^{2\pi i/7})$.

4. Determine the minimal polynomial of $\cos 2\pi/7$ over $\mathbb{Q}$.

5. If $n$ is an odd positive integer, prove that
   $$\mathbb{Q}(\cos(2\pi/n)) = \mathbb{Q}(\cos(\pi/n)).$$

6. Take a regular $n$-sided polygon inscribed in a circle of radius 1. Label the vertices $P_1, \ldots, P_n$, and let $\lambda_k$ be the length of the line joining $P_k$ and $P_n$ for $1 \leq k \leq n - 1$. Prove that
   $$\lambda_1 \cdots \lambda_{n-1} = n.$$

7. Find a real number $\alpha$ such that the extension $\mathbb{Q} \subset \mathbb{Q}(\alpha)$ is Galois, with Galois group $\mathbb{Z}/6$.

8. If $a$ and $b$ are rational numbers satisfying $a^2 + b^2 = 1$, use Hilbert’s Theorem 90 to prove that
   $$a = \frac{s^2 - t^2}{s^2 + t^2} \quad \text{and} \quad b = \frac{2st}{s^2 + t^2} \quad \text{for some } s, t \in \mathbb{Q}.$$

   This shows that any right triangle with integer sides has sides of length
   $$d(s^2 - t^2), \quad 2dst, \quad d(s^2 + t^2), \quad \text{for } d, s, t \in \mathbb{Z}.$$

9. Let $\sigma$ and $\tau$ be automorphisms of the field $\mathbb{C}(x)$ that fix $\mathbb{C}$ and satisfy
   $$\sigma(x) = e^{2\pi i/3}x, \quad \tau(x) = 1/x.$$

   Prove that $\langle \sigma, \tau \rangle$ is isomorphic to the symmetric group $S_3$, and determine the fixed field.

10. Prove that $\mathbb{F}_4(x)$ is Galois over $\mathbb{F}_4(x^4 + x)$, and compute the Galois group. Determine the intermediate fields.