#13

Prove: $u(x, t+\frac{L}{c}) = -u(x,t)$

$f(L-x) = f(x)$

$g(L-x) = g(x)$

Proof

$$u(x,t) = \underbrace{\frac{1}{2}\left(f^*(x+ct) + f^*(x-ct)\right)}_{H(x,t)} + \underbrace{\frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\,ds}_{I(x,t)}$$

You prove that $H(x, t+\frac{L}{c}) = -H(x,t)$

$$I(x, t+\frac{L}{c}) = \frac{1}{2c}\int_{x-c(t+\frac{L}{c})}^{x+c(t+\frac{L}{c})} g^*(s)\,ds$$

$$= \frac{1}{2c}\int_{x-ct-L}^{x+ct+L} g^*(s)\,ds =$$

change of variables

$u = s+L$ The upper bound on the integral

$du = ds$ ~~up~~ ~~up~~ $u_{up} = s_{up} + L = x+ct+2L$

$s = u-L$ Lower bound: $u_e = s_e + L = x-ct-L+L = x-ct$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct+2L} g^*(u)\,du = \frac{1}{2c}\int_{x-ct}^{x+ct+2L} g^*(s+L)\,ds$$

Now $g^*(s+L) = g^*(L-(-s)) = g^*(-S) = -g^*(S)$

from the symmetry property given

$$\Rightarrow -\frac{1}{2c}\int_{x-ct}^{x+ct+2L} g^*(s)\,ds =$$

$$= -\frac{1}{2c}\left[\int_{a}^{x+ct+2L} g^*(s)\,ds - \int_{a}^{x-ct} g^*(s)\,ds\right]$$

$$= -\frac{1}{2c}\left[G(x+ct+2L) - G(x-ct)\right]$$

Since G is $2L$-periodic

$$= -\frac{1}{2c}\left[G(x+ct) - G(x-ct)\right] = -\frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\,ds$$

$$= -I(x,t)$$

\#14

I will prove that $u(L,t)=0$

The rest is yours to prove

$$u(x,t)=\frac{1}{2}\left[f^*(x+ct)+f^*(x-ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\,ds$$

$$u(L,t)=\underbrace{\frac{1}{2}\left[f^*(L+ct)+f^*(L-ct)\right]}_{H(L,t)}+\underbrace{\frac{1}{2c}\int_{L-ct}^{L+ct} g^*(s)\,ds}_{I(L,t)}$$

$$H(L,t)=\frac{1}{2}\left[f^*(L+ct)+f^*(L-ct)\right]$$

|| since f^* is $2L$-periodic

$$=\frac{1}{2}\left[f^*(L+ct-2L)+f^*(L-ct)\right]$$

$$=\frac{1}{2}\left[f^*(ct-L)+f^*(L-ct)\right]=$$

↕ since f^* is odd

$$=\frac{1}{2}\left[-f^*(L-ct)+f^*(L-ct)\right]=0$$

$$I(t,t) = \frac{1}{2c}\int_{L-ct}^{L+ct} g^*(s)\,ds =$$

$$= \frac{1}{2c}\left[G(L+ct) - G(L-ct)\right] \Rightarrow$$

G is 2L periodic

$G(x-2L) = G(x)$

$$= \frac{1}{2c}\left[G(L+ct-2L) - G(L-ct)\right] \Rightarrow$$

$$= \frac{1}{2c}\int_{L-ct}^{ct-L} g^*(s)\,ds = \frac{1}{2c}\int_{L-ct}^{-(L-ct)} g^*(s)\,ds = 0$$

Since $g^*(x)$ is odd integrated over

$\forall$

$$\Rightarrow u(L,t) = 0$$