RESEARCH STATEMENT

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My area of research is the applied mathematics of composite materials, and in particular, problems in effective properties of composites, homogenization, electromagnetic scattering and diffraction, wave propagation in heterogeneous media, optimal design problems, photonic device modelling, metamaterials, and plasmonics.

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1. AN EFFECTIVE COMPLEX PERMITTIVITY FOR TIME HARMONIC WAVES IN RANDOM MEDIA.

1.1. Background. Usually, when one considers the propagation of an electromagnetic wave in a random medium, two length scales are of importance. The first scale is the wavelength λ of the electromagnetic wave probing the medium. The second one is the typical scale of the inhomogeneities δ. There has been plenty of work to build an effective medium theory that is applicable to wave propagation and fields that oscillate with time provided that the wavelengths associated with the fields are much larger than the microstructure. This limit where the size of the microstructure goes to zero is called the quasistatic or infinite wavelength limit. In this case the heterogeneous material is replaced by a homogeneous fictitious one whose macroscopic characteristics are good approximations of the initial ones. The solutions of a boundary value partial differential equation describing the propagation of waves converge to the solution of a limit boundary value problem which is explicitly described when the size of the heterogeneities goes to zero.

The problem of finding bounds on the effective properties of materials in the quasistatic limit has been investigated vigorously, and there have been significant advances not only in deriving optimal bounds, but also in describing the materials that accomplish these bounds [16] and references within. Wellander and Kristensson [26] and Conca and Vanninathan [2] have both recently analyzed the homogenization of time-harmonic wave problems in periodic media. Their results are each applicable to problems in which the wavelength of the incident field is much larger than the microstructure.

For waves in random media, Keller and Karal [10] and Papanicolaou [21] use averaging of random realizations of materials in order to describe the effective properties of the composites when interacting with electromagnetic waves. Both analyses assume that the random materials deviate slightly from a homogeneous material, i.e. the contrast of the random inclusions is small. Keller and Karal assume *a priori* that the effective dielectric coefficient is a constant. Using perturbation methods, they approximate the dielectric constant with a complex number, whose imaginary part accounts for the wave attenuation.

1.2. Previous Research. The above methods that provide bounds and describe the behavior of the dielectric coefficients do not account for scattering effects which occur when the wavelength is no longer much larger than the inhomogeneities of the composite and when the contrast is large. This problem has remained open and results are sparse. The problem is difficult and none of the techniques that come from the quasistatic regime can be applied directly to the scattering problem since all of the quasistatic methods utilize the condition that the size of the heterogeneities goes to zero.

Even the correct definition of "effective medium" is somewhat unclear outside the quasistatic regime. In this work, we assume that the purpose of the effective medium is to reproduce the average or expected wave field as the actual medium varies over a given set of random realizations. In the quasistatic case, the effective permittivity ε^* is defined by

$$\varepsilon^* \langle E \rangle = \langle D \rangle = \langle \varepsilon E \rangle,$$

where the averaged electric field $\langle E \rangle = \bar{E}$ is a given constant, and the averaged dielectric displacement $\langle D \rangle$ is independent of x which ensures that ε^* in the quasistatic case is a constant.

For simplicity in this work we consider waves in 2- or 3-dimensional random media governed by the Helmholtz equation

$$\triangle u + \omega^2 \varepsilon u = 0,$$

where the permittivity function $\varepsilon(x)$ assumes random realizations within some probability space. We average over all the possible material realizations to obtain the equation

$$\triangle \langle u \rangle + \omega^2 \langle \varepsilon u \rangle = f,$$

where $\langle \rangle$ denotes expected value, i.e. averaging over the set of realizations, and not a spatial average. We seek to find the dielectric coefficient ε^* that will solve the problem

$$\triangle \langle u \rangle + \omega^2 \varepsilon^* \langle u \rangle = f, \tag{1.1}$$

where $\langle u \rangle$ is the averaged solution. From the above two equations, it is easy to see that the appropriate definition for ε^* is

$$\varepsilon^* = \frac{\langle \varepsilon u \rangle}{\langle u \rangle}. \tag{1.2}$$

Note that the definition of ε^* does not preclude spatial variations $\varepsilon^* = \varepsilon^*(x)$.

Wave localization and cancellation must be accounted for when the wavelength is in the same order as the size of the heterogeneities, which means that the effective coefficients are no longer necessarily constants as in the quasistatic case, but functions of the space variable. We have illustrated in [24] that as ω increases (which will decrease the wavelength), we begin to see spatial variations in the effective dielectric coefficient due to the presence of scattering effects. Nevertheless ε^* as defined in (1.2) is a "correct" definition of the effective dielectric coefficient, in that it reproduces the average field response through equation (1.1).

Since ε^* cannot be calculated explicitly in general, to be useful in applications it is important that we can bound both ε^* itself, and some measure of the spatial variations in ε^*. A bound on the magnitude of ε^* and a bound on the total variation, $\|\varepsilon^*\|_{BV}$ are presented in [4]. The estimates hold for any fixed frequency $\omega > 0$ and show an explicit dependence on the feature size and contrast of the random medium:

THEOREM 1.1. *For sufficiently small δ, $\varepsilon^*(x)$ is bounded from above uniformly for all $x \in \Omega$. For such δ there exists a constant C^* such that $\|\varepsilon^*\|_{BV} \leq C^*|\varepsilon_1 - \varepsilon_0|\delta$, and the spatial variations of $\varepsilon^*(x)$ are bounded in terms of the size of the inhomogeneities δ and the contrast of the medium $|\varepsilon_1 - \varepsilon_0|$. As the size of the inhomogeneities goes to 0, the spatial variations decrease in magnitude, and $\varepsilon^*(x) \to p\varepsilon_0 + (1-p)\varepsilon_1$.*

The main theorem is proven by combining probability arguments with the regularity properties of the solutions. Pertinent numerical experiments are performed to illustrate the results of the analytical proof.

A problem of finding the class of materials (described by a probability density function $h(\psi)$) that maximizes the spatial average of the dielectric coefficient is presented:

$$\begin{aligned}
\max_h \int_\Omega \frac{\int_{\Psi_\delta} \varepsilon u h\, dP}{\int_{\Psi_\delta} u h\, dP} dx \qquad & \text{subject to} \\
\int_{\Psi_\delta} h\, dP = 1; & \\
h(\mathbf{d}) \geq 0 \quad \text{for all } \mathbf{d}; & \\
\sum_i d_i = |\Omega|. &
\end{aligned}$$

Existence and uniqueness of a maximizing probability density function h_0 is proven. Numerical experiments are performed to find the maximizing probability density function.

The dependence of the effective dielectric coefficient on the contrast in the medium is explored, and series expansion of the effective coefficient that takes into account the correlation functions of the medium is derived:

$$\varepsilon^* = 1 + z\frac{pq - z\omega^2\langle\chi A_\omega \chi\rangle q + z^2\omega^4\langle\chi A_\omega \chi A_\omega \chi\rangle q - \ldots}{q - zp\omega^2 A_\omega q + z^2\omega^4 A_\omega \langle\chi A_\omega \chi\rangle q - \ldots},$$

where z is the contrast in the medium, $q = g_\omega \star f$ with g_ω being the free-space Green's function for the operator $Lv = \triangle v + \omega^2 v$ (with the outgoing wave condition) and f - the incident field. A_ω is Lippmann-Schwinger operator, and the function $\chi(x,\psi)$ is a random characteristic function in x. The terms of form $\langle\chi A_\omega \chi\rangle q$, $\langle\chi A_\omega \chi A_\omega \chi\rangle q$, etc, are calculated explicitly using Keller and Karal's [10] suggestion of applying the mean value theorem for the solution of constant coefficient Helmholtz equation. Every term in the series is a constant provided the medium is stationary. We approximate the effective dielectric coefficient for media with a correlation function depending on position, and discover that the best approximation is a function of the space variable.

1.3. Related Future Research. We have proven the analyticity of the effective dielectric coefficient (1.2) with respect to the contrast in a certain region, but we have not investigated the analytical properties of the effective dielectric coefficient thoroughly, and questions about extending the region of analyticity, and obtaining bounds using analytical methods have not been approached.

We would like to extend our results in describing the effective properties of composite materials to the TM-polarization case where the magnetic field $H(x_1, x_2, x_3) =$

$(0, 0, u(x_1, x_2))$ is modeled by the equation $\nabla \cdot \left(\frac{1}{\varepsilon}\nabla u\right) + \omega^2 u = f$. In this case the spatially-dependent effective dielectric coefficient is

$$\varepsilon^*_{TM} = \frac{\nabla\langle u\rangle}{\langle\frac{1}{\varepsilon}\nabla u\rangle}.$$

Here the argument has to be modified because of the reduced regularity of the solutions.

2. OPTIMIZATION OF PERIODIC COMPOSITE STRUCTURES FOR SUB-WAVELENGTH FOCUSING.

2.1. Background. Recently, there has been a renewed and avid interest in studying a class of materials known as the left-handed materials (LHMs). These materials have simultaneously negative real parts of dielectric permittivity ρ and magnetic permeability μ, so that their refractive index is negative. The properties of such materials were investigated first by Veselago in 1967 [27]. As shown by Veselago, LHMs exhibit some peculiar electromagnetic properties such as negative index of refraction and wave vector, $\mathbf{k}$, and Poynting vector, $\mathbf{S}$, having opposite directions. Veselago realized that a slab of LHM would act as a lens.

According to Abbe's diffraction limit, conventional lenses based on positive index materials with curved surfaces are not able to resolve an object's fine details that are smaller than half of the light wavelength λ. The limitation occurs because the waves with transverse wave numbers larger than $2\pi n/\lambda$, which carry information about the fine sub-λ details of the object, decay exponentially in free space. In a negative index material slab, however, the evanescent wave components can grow exponentially and thus compensate for the exponential decay. Therefore, under ideal conditions, all Fourier components from the object can be recovered at the image plane producing a resolution far below the diffraction limit [23].

Milton, Nicorovici, McPhedran and Podolskiy proved superlensing in the quasistatic regime (where the wavelength is much larger than the object), and discussed limitations of superlenses in this regime due to anomalous localized resonance. If the source being imaged responds to an applied field, it must lie outside the resonant regions to be successfully imaged [19].

In the electrostatic limit, the magnetic and electric fields decouple, and the requirement for superlensing of transverse-magnetic waves is reduced to only $\rho = -\rho_h$, where ρ_h is the permittivity of the host medium interfacing the lens [23]. An example of such near field superlens is a slab of silver in air illuminated at its surface plasmon resonance (where $\rho = -1$). Experiments with silver slabs have already shown rapid growth of evanescent waves [13], submicron imaging [18], and imaging beyond the diffraction limit [7]. A major draw-back of such near-field superlenses based on bulk metals is that they can operate only at a single frequency ω satisfying the lens condition $\rho(\omega) = -\rho_h$. Cai, Genov and Shalaev proposed a "tunable" near-field superlens made of metal-dielectric composites that can operate at any desired visible or near-infrared wavelength with the frequency controlled by the metal filling factor of the composite (here the inhomogeneities are assumed to be much smaller than the wavelength) [25].

It was shown by Efros and Pokrovsky that a two dimensional photonic crystal made from a non-magnetic dielectric has negative values of both the electric permittivity and the magnetic permeability in some frequency range [6]. The physical principles that allow negative refraction in photonic crystals arise from the dispersion characteristics of wave propagation in periodic media and are very different from those in LHMs. They also do not require both negative electric permittivity and magnetic permeability [12, 20]. The negative refraction of beams can be described by analyzing the equifrequency surface of the band structures [12, 20, 14]. If the constant-frequency contour is everywhere convex, an incoming plane wave from air will couple to a single mode that propagates into the crystal on the negative side of the boundary, and thus negative refraction in the first band is realized. Luo, Johnson, Joannopoulos and Pendry have shown all-angle negative refraction could be achieved at the lowest band of two-dimensional photonic crystals in the case of $\mathbf{S} \cdot \mathbf{k} > 0$ [14]. Such all-angle refraction is essential for superlens application. The photonic crystal not only focuses all propagating waves without limitation of finite aperture, but also amplifies at least some evanescent waves, and the unconventional imaging effects are due to the presence of additional near-field light. A perfect lens, made of left-handed materials, focuses all propagating waves and all evanescent waves. The important difference for superlensing with a photonic crystals is that only finite number of evanescent waves is amplified. This is a consequence of Bragg scattering of light to leaky photon modes [15]. The resolution of a photonic-crystal superlens at a single frequency is only limited by its surface period instead of the wavelength [15].

More recently Huang, Lu and Sridhar proposed an alternative approach to all-angle negative refraction in two-dimensional photonic crystals. By applying appropriate modifications with surface grating to the flat photonic lens, they are able to focus large and/or far way objects [9].

2.2. Previous Research. Inspired by the current research in structures that produce sub-wavelength focusing, we use derivative-based minimization techniques to produce structures that will provide sub-wavelength focus with non-magnetic materials and without the need for negative permittivities. Rather than restricting to designs based on photonic crystal structures, we allow as admissible *any* periodic composite structure (with fixed period) whose refractive index is bounded above and below by fixed constants. And rather than performing parametric optimization over a small number of variables describing the structure, we use techniques of "topology optimization" in which material distribution is completely arbitrary. We are able to obtain structured that focus a point source that is far away from the lens, and also we can obtain structures that give an image at a chosen distance from the lens. Since structures incorporating gratings are included in our admissible class, such designs will naturally arise through the optimization process if they produce the best possible image.

For simplicity, only the case of "two-dimensional" structures in E-parallel polarization is considered. The ideas here should extend to the other polarization case and the full three-dimensional problem, although there are some technical hurdles.

We describe the model problem and present a variational formulation of the Helmholtz equation in a periodic geometry using Floquet theory. The inclusion of a small amount of energy absorption in the medium allows a uniform upper bound on the norm of the electric field, independent of the particular admissible structure (and thereby preventing resonances).

The goal of optimization is to make a perfect image of the incident field $u_i = H_0^{(1)}(\omega r)$ on the opposite side of the slab $\{y < -b\}$. A "mirror image" converging at the point $(0, -(b + h_1))$ would look like $H_0^{(2)}(\omega\sqrt{x^2 + (y + b + h_1)^2})$, where $H_0^{(2)} =$

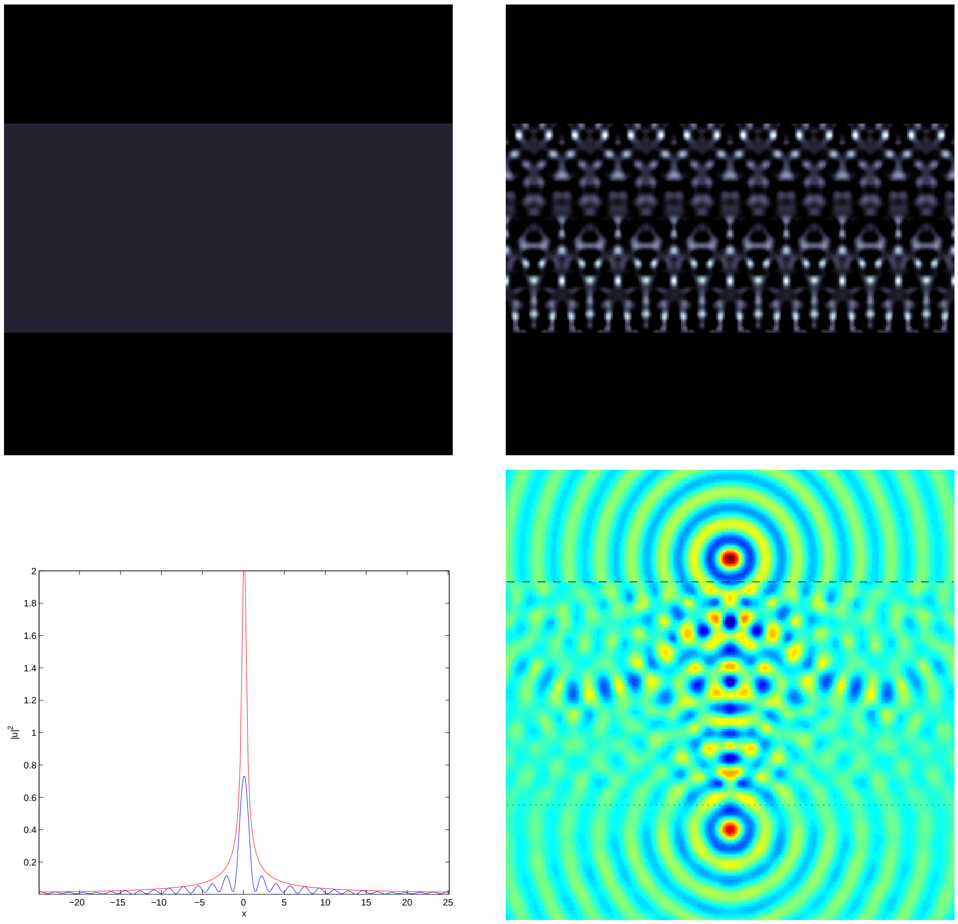

FIG. 2.1. *Results from Example 1. Upper left: initial (real) ρ; upper right: optimized solution; lower left: intensity cross section (blue) versus point source (red); lower right: real part of E field within the solution box. Eight periods are shown.*

$\overline{H_0^{(1)}}$ is the conjugate Hankel function. Thus we want the trace

$$u(x,-b) = H_0^{(2)}(\omega\sqrt{x^2+h_1^2}) \equiv q(x).$$

The Bloch representations of u allows us to see that by setting

$$u_\alpha(x,-b) = q_\alpha(x), \tag{2.1}$$

we get $u(x,-b) = q(x)$. The additional boundary condition (2.1) makes our problem overposed. However, by allowing ρ to vary as a design variable, it may be possible to make (2.1) hold approximately for each α. Let $F(\rho,\alpha) = u_\alpha|_{\Gamma_b}$, where $u_\alpha \in H^1(\Omega)$ is the weak solution to our variational problem and Γ_b is the boundary.

Consider the minimization

$$\inf_{\rho\in\mathcal{A}} J(\rho) = \frac{1}{2}\int_{-1/2}^{1/2} \|F(\rho,\alpha) - q_\alpha\|_{L^2}^2\, d\alpha.$$

Theorem 2.1. *The optimization problem has a solution.*

The proof follows the direct method in the calculus of variations. We compute the L^2-gradient of the functional $J(\rho)$ using the solution w_α of the corresponding backward problem and derive the optimality conditions.

The linear sensitivity of the design of the medium to a small perturbation can be estimated. The medium becomes less sensitive to perturbation as the lower bound on the loss ρ_{i_0} increases. Conversely, extremely sensitive designs are possible in case of small (or no) loss. The sensitive designs may correspond to locally large fields u_α, w_α which can occur near resonances.

We perform the numerical experiments using gradient descent method and show structures that have produced sub-wavelength focus, e.g. Figure 2.1. We have demonstrated an optimized structure that gives a focus with a spot size 0.284λ, which is a significant improvement to those obtained by the regular-material structures described in the research literature so far.

3. FUTURE RESEARCH.

3.1. Optimization of composite structures for constructing a regular-material optimal cloak. Another interesting optimization problem, that I am currently exploring, is finding the optimal cloak that approximately conceals an object by leaving the electromagnetic field almost undisturbed and minimizing (or ideally eliminating) the scattering by the cloaked obstacle. The cloaking outside is independent of what is inside the cloak.

Milton and Nicorovici proved the existence of a cloak made of negative-index metamaterials in the quasistatic limit for finite collections line or point dipoles [17]. Pendry, Schurig and Smith suggest a coordinate transformation approach for designing a structure made of metamaterials. When electromagnetic waves pass through the cloak, it will deflect the waves, guide them around the object, and return them to the original propagation direction without perturbing the exterior field, thereby rendering the interior effectively "invisible" to the outside [22]. In the quasistatic case, Kohn, Shen, Vogelius and Weinstein prove that the change-of-variable-based scheme achieves perfect cloaking in any dimension $n \geq 2$, and obtain a regular approximate cloak using a nonsingular change of variables [11]. Greenleaf, Kurylev, Lassas and Uhlmann provide a rigorous proof of cloaking for Helmholtz equation based on coordinate transformation [8]. There a construction for cloaking a region D contained in a domain $\Omega \subset \mathbb{R}^n, n \geq 3$ is proposed, where measurements of Cauchy data of waves on Ω are made at $\partial\Omega$. This single coating construction, providing invisibility for Helmholtz equation, requires surrounding the outer boundary ∂D^+ of the cloaked region with metamaterials whose material parameters (index of reflection, or electric permittivity and magnetic permeability) are conformal to a singular Riemannian metric on Ω, i.e. zero or infinity material parameters are required. Similar double coating for Maxwell's equations is presented.

Using regular materials one cannot achieve an ideal cloak. Our goal is to construct a cloak that will perturb the electromagnetic field minimally producing a small scattered field. The problem we propose is the following. For a fixed frequency $0 < \omega < \infty$, let the Cauchy data $\Lambda^\omega_{A,q}$ consist of the set of pairs of boundary measurements $(u|_{\partial\Omega}, \partial_\nu|_{\partial\Omega})$ where u ranges over solutions of equation $\nabla \cdot A\nabla u + \omega^2 qu = 0$ in Ω. Let $A = A_D + A_\Omega$ where $\text{supp}(A_D) \subset D$ and $\text{supp}(A_\Omega) \subset \Omega - D$, and similarly for $q = q_D + q_\Omega$. We allow the design variables A_D, A_Ω, q_D, and q_Ω to vary within an admissible class of coefficients bounded above and below by fixed constants. Define the functional

$$J(A,q) = \sup_{A_D,\, q_D} \|\Lambda^{\omega}_{A,q} - \Lambda^{\omega}_{1,1}\|,$$

that selects the object which creates the largest disturbance in the Cauchy data. This will allow cloaking for arbitrary objects. Consider the minimization

$$\inf_{A_\Omega,\, q_\Omega} J(A,q),$$

which seeks the structure of the cloak that will minimize the perturbation of the external field.

We are pursuing the problem on two fronts. First, we hope to establish the well-posedness of the problem and characterize its solutions through mathematical analysis. Second, we plan to search for approximate solutions numerically.

3.2. Plasmonics. My interest in plasmonics comes naturally since it describes the interaction of metals with electromagnetic fields in a classical framework based on Maxwell's equations. The electromagnetic properties of metal/dielectric interfaces have attracted a vast amount of recent research effort. Such structures have the ability to sustain coherent electron oscillations known as surface plasmon polaritons leading to electromagnetic fields confined to the metallic surface. These electromagnetic surface waves propagate at the interface between a dielectric and a conductor and are evanescently confined in the perpendicular direction. They arise via the coupling of the electromagnetic fields to oscillations of the conductor's electron plasma. Such structures have been intensively investigated in the physics and engineering literature, but not much research has been done in the mathematical optimization of plasmonic structures.

The propagation direction of surface plasmon polaritons at the interface of a metal film and a dielectric superstrate can be controlled via scattering of the propagating waves at locally created defects in an otherwise planar film. The scatterers can be introduced in the form of surface undulations or by milling holes into the film. Their controlled positioning enables the generation of functional elements such as Bragg mirrors for reflecting surface plasmon polaritons [3]. This concept can be extended to creating plasmon photonic crystals exhibiting band gaps in desired frequency regions. Bozhevolnyi *et al.* demonstrated that a triangular lattice of gold dots on a thin gold film establishes a band gap for surface plasmon polariton propagation [1]. The band gap can be determined by determining the penetration distance of the surface waves into the lattice structure space. Optimization techniques can be applied to creating structures that maximize the width of the band gap. An application of this concept in waveguiding is obvious: by creating micron-wide line defects where the triangular lattice of scatterers is locally removed, surface plasmon polaritons can be confined in channel waveguides, akin to well established concepts in dielectric photonic crystals.

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