#2

$$b_n = \frac{4}{\pi}\int_0^{\pi/2} f(x)\ \sin 2nx\, dx = 0$$

use trig identities!

$$b_n^* = \frac{2}{n\pi}\int_0^{\pi/2} x\cos x \sin 2nx\, dx$$

$$= \frac{2}{n\pi}\left[\int_0^{\pi/2} \frac{x}{\cancel{2}}\left(\sin[(2n+1)x] + \sin[(2n-1)x]\right) dx\right]$$

$$= \frac{\cancel{2}1}{n\pi}\left[\int_0^{\pi/2} x\sin[(2n+1)x]\, dx + \int_0^{\pi/2} x\sin[(2n-1)x]\, dx\right]$$

$$= \frac{1}{n\pi}\left[\frac{1}{(2n+1)^2}\sin(2n+1)\cancel{\pi/2}x - \frac{x}{2n+1}\cos(2n+1)x\Big|_0^{\pi/2}\right.$$

$$+ \frac{1}{(2n-1)^2}\sin(2n-1)x - \frac{x}{2n-1}\cos(2n-1)x\Big|_0^{\pi/2}$$

$$= \frac{1}{n\pi}\left[\frac{1}{(2n+1)^2}\sin(2n+1)\pi/2 - \cancel{\frac{\pi}{2(2n+1)}\cos(2n+1)\pi/2}^{\,0} - \cancel{\frac{1}{(2n+1)^2}\sin 0}^{\,0}\right.$$

$$- \cancel{\frac{0}{2n+1}\cos 0}^{\,0} + \frac{1}{(2n-1)^2}\sin(2n+1)\pi/2 \quad \Rightarrow$$

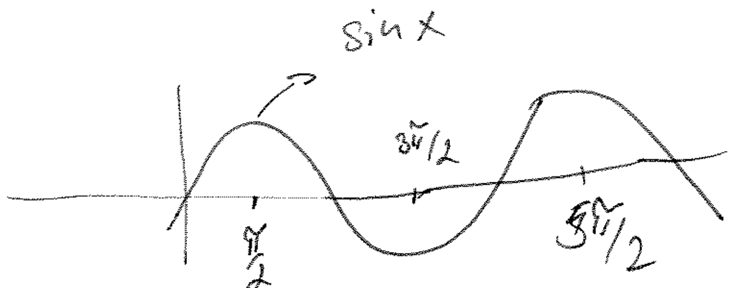
<u>Note</u>: $(2n+1)$ and $(2n-1)$ are odd integers for every n

$\Rightarrow \cos(2n+1)\pi/2 = 0$ and $\cos(2n-1)\pi/2 = 0$

$$b_n^* = \frac{+1}{n\pi}\left[\frac{(-1)^n}{(2n+1)^2} - \frac{(-1)^n}{(2n-1)^2}\right]$$

$$= \frac{(-1)^n}{n\pi}\left[\frac{1}{(2n+1)^2} - \frac{1}{(2n-1)^2}\right] \Rightarrow$$

$$= \frac{(-1)^n}{\cancel{n}\pi} * \left[\frac{\cancel{4n^2} - 4n + \cancel{1} - \cancel{4n^2} - 4n - \cancel{1}}{(4n^2-1)^2}\right] = \frac{-8(-1)^n}{\pi(4n^2-1)^2}$$



$$\Rightarrow \sin\left[\underbrace{(2n+1)}_{\text{odd}}\frac{\pi}{2}\right] = \begin{cases} 1 & \text{when } n\text{-even} \\ -1 & \text{when } n\text{-odd} \end{cases} = (-1)^n$$

$$\sin\left[(2n-1)\frac{\pi}{2}\right] = \begin{cases} -1 & \text{when } n\text{-even} \\ 1 & n\text{-odd} \end{cases} = -(-1)^n$$

$$\Rightarrow u(x,t) = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(4n^2-1)^2} \sin 2nt \sin 2nt$$

II #3 $\quad L=\pi \quad c=1 \quad T_1=100 \quad T_2=50$

$$u_{steady} = \frac{T_2-T_1}{L}x + T_1 = \frac{50-100}{\pi}x + 100$$

$$= -\frac{50}{\pi}x + 100$$

$$u_{oscillatory} = \sum b_n e^{-\lambda_n^2 t} \sin\frac{n\pi x}{L}$$

$$b_n = \frac{2}{\pi}\int_0^\pi \left[f(x) - \left(-\frac{50x}{\pi} + 100\right)\right] \sin nx \, dx$$

$$= \frac{2}{\pi} I_1 - \frac{2}{\pi} I_2$$

$$\Rightarrow u_{oscillatory} = \sum_{k=0}^{\infty} \frac{+132(-1)^k}{\pi(2k+1)^2} e^{-(2k+1)^2 t} \sin(2k+1)$$

$$- \frac{100}{\pi} \sum_{n=1}^{\infty} \underbrace{\frac{2\left(1 - \frac{(-1)^n}{2}\right)}{n}}_{= \frac{2-(-1)^n}{n}} e^{-n^2 t} \sin nx$$

$u = u_{steady} + u_{oscillatory}$

$$= \int_0^{\pi} f(x)\sin nx\,dx = 33\int_0^{\pi/2} x\sin nx\,dx + 33\int_{\pi/2}^{\pi} (\pi - x)\sin nx\,dx$$

$$= 33\left[\frac{1}{n^2}\sin nx - \frac{x}{n}\cos nx\Big|_0^{\pi/2}\right.$$

$$-\frac{\pi}{n}\cos nx\Big|_{\pi/2}^{\pi}$$

$$\left.-\frac{1}{n^2}\sin nx + \frac{x}{n}\cos nx\Big|_{\pi/2}^{\pi}\right]$$

$$= 33\left[\frac{1}{n^2}\sin\frac{n\pi}{2} - \frac{\pi}{2n}\cancel{\cos n\pi/2} - \cancel{\frac{\pi}{n}\cos n\pi} + \frac{\pi}{n}\cancel{\cos\frac{n\pi}{2}}\right.$$

$$\left.+\cancel{\frac{\pi}{n}\cos n\pi} - \frac{\pi}{2n}\cancel{\cos\frac{n\pi}{2}} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right.$$

$$= \frac{66}{n^2}\sin\frac{n\pi}{2} = \begin{cases} 0 & \text{when } n\text{-even} \\ \dfrac{66}{(2k+1)^2} & \text{when } n\text{-odd},\ k\text{-even} \\ -\dfrac{66}{(2k+1)^2} & n\text{-odd},\ k\text{-odd} \end{cases}$$

$$\lambda = \int_0^{\pi} -\frac{50x}{\pi}\sin nx\,\cancel{dx} + \int_0^{\pi} 100\sin nx\,dx =$$

$$= -\frac{50}{\pi}\left[\frac{1}{n^2}\cancel{\sin nx}^{\,0} - \frac{x}{n}\cos nx \Big|_0^{\pi}\right] + \frac{-100}{n}\cos nx \Big|_0^{\pi}$$

$$= \frac{50}{n}\cos n\pi - \frac{100}{n}\cos n\pi + \frac{100}{n} = -\frac{50(-1)^n}{n} + \frac{100}{n}$$

$$= \frac{100}{}\left(\frac{1 - \frac{(-1)^n}{2}}{n}\right)$$

$\#3$ $\boxed{\text{III}}$

$$B_{mn} = 4 \int_0^1 \int_0^1 \sin \pi x \sin 3\pi y \sin m\pi x \sin n\pi y \, dx \, dy$$

$$= 4 \underbrace{\left(\int_0^1 \sin \pi x \sin m\pi x \, dx \right)}_{= \begin{cases} 0 & \text{when } m \neq 1 \\ \frac{1}{2} & m = 1 \end{cases}} \underbrace{\left(\int_0^1 \sin 3\pi y \sin n\pi y \, dy \right)}_{= \begin{cases} 0 & \text{when } n \neq 3 \\ \frac{1}{2} & n = 3 \end{cases}}$$

$$= \begin{cases} 0 & m \neq 1 \text{ or } n \neq 3 \\ 1 & m = 1 \quad n = 3 \end{cases}$$

$$\Rightarrow \lambda_{13} = \pi \frac{1}{\pi} \sqrt{1 + 3^2} = \sqrt{10}$$

$\boxed{\text{III} \quad \# 3}$

$$B^*_{mn} = \frac{4}{(1)(1)\lambda_{mn}} \int_0^1 \int_0^1 \cancel{g(x,y)}\; 1 \sin\frac{m\pi x}{1} \sin n\pi y \, dx\, dy$$

$$= \frac{4}{\lambda_{mn}} \left(\int_0^1 \sin m\pi x \, dx \right) \left(\int_0^1 \sin n\pi y \, dy \right)$$

$$= \begin{cases} 0 & \text{when } n \text{ or } m \text{ even} \\ \dfrac{16}{\lambda_{mn}\pi^2} \dfrac{1}{mn} & \text{when } n \text{ and } m \text{ - odd} \end{cases}$$

$$\Rightarrow \begin{matrix} m = 2k+1 \\ n = 2l+1 \end{matrix} \quad \text{and} \quad \lambda_{mn} = \sqrt{(2k+1)^2 + (2l+1)^2}$$

$$\Rightarrow u(x,y,t) = \sin \pi x \sin 3\pi y \cos \sqrt{10}\, t + \frac{16}{\pi^2} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\sin (2k+1)\pi x \sin (2l+1)\pi y}{(2k+1)(2l+1)\sqrt{(2k+1)^2 + (2l+1)^2}} \times \sin\left(\sqrt{(2k+1)^2 + (2l+1)^2}\; t\right)$$

$$\int_0^1 \sin m\pi x \, dx = \frac{-1}{m\pi} \cos m\pi x \Big|_0^1 = \frac{-(-1)^m + 1}{m\pi}$$

$$= \begin{cases} 0 & \text{when } m \text{ - even} \\ \dfrac{2}{m\pi} & \text{when } m \text{ - odd} \end{cases}$$

similarly $\displaystyle \int_0^1 \sin n\pi y \, dy = \begin{cases} 0 & \text{when } n \text{ - even} \\ \dfrac{2}{n\pi} & \text{when } n \text{ - odd} \end{cases}$

#2

$f(x,y) = \sin \pi x \sin \pi y$

$$A_{mn} = \frac{4}{(1)(1)} \int_0^1 \int_0^1 \sin \pi x \sin \pi y \sin m\pi x \sin n\pi y \, dx\, dy$$

$$= 4 \left(\int_0^1 \sin \pi x \sin m\pi x \, dx \right) \left(\int_0^1 \sin \pi y \sin n\pi y \, dy \right)$$

$$= \begin{cases} 0 & \text{when } m \neq 1 \text{ or } n \neq 1 \\ 1 & \text{when } m = 1 \text{ and } n = 1 \end{cases}$$

$\Rightarrow \lambda_{11} = \pi \sqrt{2}$

$\Rightarrow u(x,y,t) = \sin \pi x \sin \pi y \, e^{-2\pi^2 t}$

$$\int_0^1 \sin \pi x \sin m\pi x \, dx = \frac{\sin[(m-1)\pi x]}{2(m-1)\pi} - \frac{\sin[(m+1)\pi x]}{2(m+1)\pi} \Bigg|_0^1$$

$= 0$ when $m \neq 1$

$$\int_0^1 \sin^2 \pi x \, dx = \int_0^1 \frac{1-\cos 2\pi x}{2} dx = \frac{1}{2} x - \frac{1}{4\pi} \sin 2\pi x \Bigg|_0^1$$

$= \frac{1}{2}$ when $m = 1$

Similarly, the integral $\int \cdots dy = \begin{cases} 0 & \text{when } n \neq 1 \\ \frac{1}{2} & \text{when } n = \frac{1}{2} \end{cases}$