Sample Problems Solutions

#1: $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

$$\frac{\partial u}{\partial x} + \frac{y}{x}\frac{\partial u}{\partial y} = 0$$

$$p(x,y) = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{1}{y}\,dy = \frac{1}{x}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\ln y = \ln x + C$$

$$e^{\ln y} = e^{\ln x + C}$$

$y = C x \Rightarrow$ characteristic curves: $C = \frac{y}{x}$

$\Rightarrow u(x,y) = F\left(\frac{y}{x}\right)$

where F is any diff. func. of one variable

check:

$$\frac{\partial u}{\partial x} = (F')\left(-\frac{y}{x^2}\right)$$

$$\frac{\partial u}{\partial y} = (F')\left(\frac{1}{x}\right)$$

#4:

$$(x^3+1)^3 \frac{\partial u}{\partial x} + x^2 \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} + \frac{x^2}{(x^3+1)^3} \frac{\partial u}{\partial y} = 0$$

$$p(x,y) = \frac{x^2}{(x^3+1)^3}$$

$$\frac{dy}{dx} = \frac{x^2}{(x^3+1)^3}$$

$$dy = \frac{x^2}{(x^3+1)^3} dx$$

$$\int dy = \int (x^3+1)^{-3} x^2 dx$$

$v = x^3+1$ $\qquad \frac{1}{3}\int v^{-3} dv = -\frac{1}{6} v^{-2}$

$dv = 3x^2 dx$

$$y = -\frac{1}{6}(x^3+1)^{-2} + C$$

$$y + \frac{1}{6}(x^3+1)^{-2} = C \quad \text{(characteristic curves)}$$

$$u(x,y) = F\left(y + \frac{1}{6}(x^3+1)^{-2}\right)$$

5a)

$$\frac{\partial u}{\partial t} = c F'(x+ct) - c G'(x-ct)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 F''(x+ct) + c^2 G''(x-ct)$$

$$\frac{\partial u}{\partial x} = F'(x+ct) + G'(x-ct)$$

$$\frac{\partial^2 u}{\partial x^2} = F''(x+ct) + G''(x-ct)$$

$$\Rightarrow u(x,t) = F(x+ct) + G(x-ct)$$

is a solution to 1-d wave eqn.

5b (*)

$$\frac{\partial u}{\partial t} = cF'(x+ct) - cG'(x-ct)$$

$$\frac{\partial u}{\partial t}(x,0) = cF'(x) - cG'(x) = 0$$

const.

$$\Rightarrow F'(x) = G'(x) \Rightarrow F(x) = G(x) + K$$

$$u(x,0) = F(x) + G(x) = 2G(x) + K = e^{-x^2}$$

$$\Rightarrow K = 0 \text{ and } G(x) = \frac{e^{-x^2}}{2} = F(x)$$

$$\Rightarrow u(x,t) = \frac{e^{-(x+ct)^2}}{2} + \frac{e^{-(x-ct)^2}}{2}$$

$\boxed{5b\ (**)}$

$u(x,0)=0$

$$\frac{\partial u}{\partial t}(x,0) = -2x e^{-x^2}$$

$$\Rightarrow F(x)+G(x)=0 \Rightarrow F(x) = -G(x)$$

$$F'(x+ct)c + G'(x-ct)(-c) = \frac{\partial u}{\partial t}(x,t)$$

$$cF'(x) - cG'(x) = -2x e^{-x^2}$$

$$F'(x) - G'(x) = -\frac{2}{c} x e^{-x^2}$$

But since $F(x) = -G(x) \Rightarrow$

$$\Rightarrow F'(x) = -G'(x)$$

$$\Rightarrow 2F'(x) = -\frac{2}{c} x e^{-x^2}$$

$$F'(x) = -\frac{1}{c} x e^{-x^2}$$

$$\Rightarrow F(x) = -\frac{1}{c}\int_0^x s e^{-s^2}\,ds =$$

$$= +\frac{1}{2c} e^{-s^2}\Big|_0^x = +\frac{1}{2c} e^{-x^2} - \frac{1}{2c} = -G(x)$$

$$\Rightarrow u(x,t) = +\frac{1}{2c} e^{-(x+ct)^2} \cancel{-\frac{1}{2c}} - \frac{1}{2c} e^{-(x-ct)^2} \cancel{+\frac{1}{2c}}$$

(* * *)

$$u(x,0) = F(x) + G(x) = 0$$

$$\Rightarrow F(x) = -G(x)$$

$$\Rightarrow F'(x) = -G'(x)$$

$$\frac{\partial u}{\partial t}(x,0) = cF'(x) - cG'(x) = \frac{x}{(1+x^2)^2}$$

$$\Rightarrow 2cF'(x) = \frac{x}{(1+x^2)^2}$$

$$F'(x) = \frac{1}{2c}\frac{x}{(1+x^2)^2}$$

$$F(x) = \frac{1}{2c}\int_0^x \frac{s}{(1+s^2)^2}\,ds \qquad \begin{aligned} v &= 1+s^2 \\ dv &= 2s\,ds \end{aligned}$$

$$= \frac{1}{4c}\int v^{-2}\,dv = -\frac{1}{4c}v^{-1} = -\frac{1}{4c}\frac{1}{(1+s^2)}\Big|_0^x$$

$$= -\frac{1}{4c}\frac{1}{1+x^2} + \frac{1}{4c} = -G(x)$$

$$u(x,t) = -\frac{1}{4c}\frac{1}{1+(x+ct)^2} + \cancel{\frac{1}{4c}} + \frac{1}{4c}\frac{1}{1+(x-ct)^2} - \cancel{\frac{1}{4c}}$$

$$= -\frac{1}{4c}\frac{1}{1+(x+ct)^2} + \frac{1}{4c}\frac{1}{1+(x-ct)^2}$$