Dear students,

7.3 ex 1 and 5 are a bit tricky and use transforms of some known functions + result of Ex. 10 in 7.2

I will show you how $\hat{u}(x,0)$ is found. You will <u>not</u> have anything that involves functions ~~that~~ whose Fourier Transforms need similar manipulations on the Final.

7.3 #1

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\hat{f}(w) \cos wt + \frac{1}{cw} \hat{g}(w) \sin wt \right] e^{iwx} \, dx$$

where $f(x) = \frac{1}{1+x^2}$ and $g(x) = 0$

and $\hat{f}$ and $\hat{g}$ are their Fourier Transforms

How do you find $\hat{f}$. It is quite tricky, and you do not need to know this technique for the final. You use the Fourier series of a different function and the result of Ex. 10b in 7.2:

1st observe:

If $g(x) = \begin{cases} e^{-x} & x>0 \\ e^{x} & x<0 \end{cases}$

then $$\hat{g}(w) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{x} e^{-iwx}\,dx + \int_{0}^{\infty} e^{-x} e^{-iwx}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{x(1-iw)}\,dx + \int_{0}^{\infty} e^{-x(1+iw)}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{1-iw} e^{(1-iw)x}\Big|_{-\infty}^{0} + \frac{1}{-(1+iw)} e^{-x(1+iw)}\Big|_{0}^{\infty}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{1-iw} - 0 + 0 + \frac{1}{1+iw}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{1-iw} + \frac{1}{1+iw}\right] = \frac{1}{\sqrt{2\pi}}\left[\frac{1+iw+1-iw}{(1-iw)(1+iw)}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2}{1-(iw)^2}\right] = \sqrt{\frac{2}{\pi}}\,\frac{1}{1+w^2}$$

Now substitute x for w, to find

$$\frac{\sqrt{2}}{\sqrt{\pi}} f(x) = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{1}{1+x^2} = \mathcal{F}(g)(x)$$

or $\sqrt{\frac{2}{\pi}}\, f(x) = \mathcal{F}(g)(x)$

or $f(x) = \sqrt{\frac{\pi}{2}}\, \mathcal{F}(g)(x)$

or by linearity

$$f(x) = \mathcal{F}\left(\sqrt{\frac{\pi}{2}}\, g\right)(x)$$

Now take the Fourier transform of both sides:

$$\mathcal{F}(f)(w) = \mathcal{F}\left(\mathcal{F}\left(\sqrt{\frac{\pi}{2}}\, g\right)(x)\right)(w)$$

Now, we apply the result of Ex. 10 b in Sec. 7.2 (p. 407) to the RHS.

we get:

$$\tilde{\mathcal{F}}\left(\tilde{\mathcal{F}}\left(\sqrt{\frac{\pi}{2}}\, g\right)(x)\right)(w) = \sqrt{\frac{\pi}{2}}\, g(-w)$$

$$= \sqrt{\frac{\pi}{2}} \begin{cases} e^{-(-w)} & \text{if } -w > 0 \\ e^{-w} & \text{if } -w < 0 \end{cases}$$

$$= \sqrt{\frac{\pi}{2}} \begin{cases} e^{w} & \text{if } w < 0 \\ e^{-w} & \text{if } w > 0 \end{cases} = \sqrt{\frac{\pi}{2}}\, e^{-|w|}$$

$$= \tilde{\mathcal{F}}(f)(w)$$

Thus $\hat{f}(w) = \sqrt{\frac{\pi}{2}}\, e^{-|w|}$

$\hat{g}(w) = 0$ since $g(x) = 0$ and

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} e^{-|w|} \cos wt\, e^{iwx}\, dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{-|w|} \cos wt\, e^{iwx}\, dx$$

7.3 #5

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\hat{f}(w) \cos cwt + \frac{1}{cw} \hat{g}(w) \sin cwt\right] e^{iwx} \, dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} \frac{\sin x}{x}$$

we must find $\hat{f}(w)$.

Here is how you do that:

The function $f(x)$ is the Fourier transform of the Function $g(x) = \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1 \end{cases}$

See p. 399 in book (with w replaced by x).

Thus we have $f(x) = \mathcal{F}(g)(x)$

Now take Fourier transform on both sides

$$\tilde{\mathcal{F}}(f(x))(w) = \tilde{\mathcal{F}}\big(\tilde{\mathcal{F}}(g)(x)\big)(w)$$

Now we apply result from ex. 10 b in 7.2.

to the RHS:

$$\tilde{\mathcal{F}}\big(\tilde{\mathcal{F}}(g)(x)\big)(w) = g(-w) = \begin{cases} 1 & \text{if } |w| \le 1 \\ 0 & \text{if } |w| > 1 \end{cases}$$

$$= \tilde{\mathcal{F}}(f(x))(w) = \hat{f}$$

so $\hat{f} = \begin{cases} 1 & \text{if } |w| \le 1 \\ 0 & \text{if } |w| > 1 \end{cases}$

and $\hat{g} = 0$ since $g(x) = 0$

Thus,

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \cos cwt \, e^{iwx} \, dx$$