Midterm 3 Review Solutions

I. a)

$f'(x) = -15x^4 + 15x^2 = 15x^2(-x^2+1)$

$= 15x^2(1-x)(1+x)$

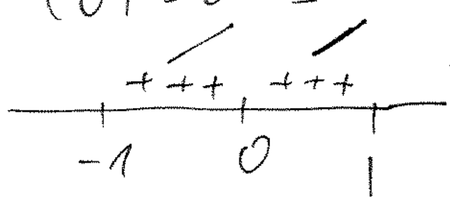
$x=0,\quad x=1,\quad x=-1$

$f'' = -60x^3 + 30x$

$f''(1) = -30$ concave down $\Rightarrow$ $x=1$ is rel. max

$f''(-1) = 60-30 = 30 \Rightarrow$ $\Rightarrow x=-1$ is a rel. min.

$f''(0) = 0 \Rightarrow$ inconclusive



neither max nor min

c) $f(x) = x + \frac{4}{x}$

$f'(x) = 1 - \frac{4}{x^2} \Rightarrow 1 - \frac{4}{x^2} = 0 \Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

$f''(x) = +\frac{8}{x^3}$

Critical points: $x = \pm 2, 0$

but $x=0$ is a vertical asymp.

$f(x)$ is not defined @ $x=0$

$f''(+2) = 1$

$\Rightarrow$ $\Rightarrow x=2$ is a min

$f''(-2) = -1$

$\Rightarrow x=-2$ is a max

d) $f(x) = \frac{x}{x^2-1} = \frac{x}{(x-1)(x+1)}$

$$f'(x) = \frac{(x^2-1) - x(2x)}{(x^2-1)^2} = \frac{x^2-1-2x^2}{(x-1)^2(x+1)^2}$$

$$= -\frac{x^2+1}{(x-1)^2(x+1)^2}$$

Critical points $x = \pm 1$, but they are V. Asymptotes

=> No relative extrema

$4-x^2 \geq 0$ $\quad - \quad + \quad -$ (-2, 2)

Domain $\Rightarrow x \in [-2, 2]$

e) $f(x) = \sqrt{4-x^2}$

$$f'(x) = \frac{1}{2}\frac{(-2x)}{\sqrt{4-x^2}} = -\frac{x}{\sqrt{(2-x)(2+x)}}$$

Critical points $x = 0, \pm 2$

$$f''(x) = \frac{\sqrt{4-x^2}\,(-1) - (-x)\frac{1}{2}\frac{-2x}{(4-x)^2}}{4-x^2}$$

$$f''(x) = -\frac{1}{\sqrt{4-x^2}} - \frac{x^2}{(4-x^2)^3}$$

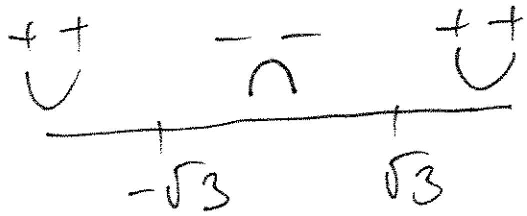
$f''(0) = -\frac{1}{2}$ $\quad - \; - \;(0)\quad$ => $x=0$ is a rel. max.

$f''(2)$ => undefined =>

$x = \mp 2$ are the endpoints of the domain

II b) $f'(x) = 4x^3 - 36x$

$f''(x) = 12x^2 - 36 = 12(x^2 - 3) = 12(x - \sqrt{3})(x + \sqrt{3})$



inflection point

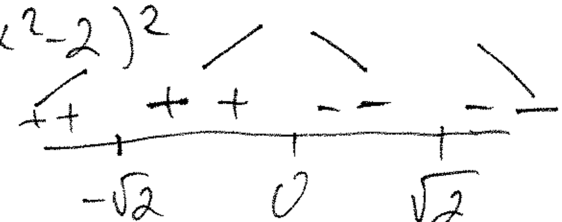
III a) $y = \dfrac{x^2+1}{x^2-2}$

H.A.: $y = 1$ y-intercept: $-\frac{1}{2}$

V.A. $x = \pm\sqrt{2}$ x-intercept: none

$$y' = \frac{(x^2-2)(2x) - (x^2+1)(2x)}{(x^2-2)^2} = \frac{2x^3 - 4x - 2x^3 - 2x}{(x^2-2)^2}$$

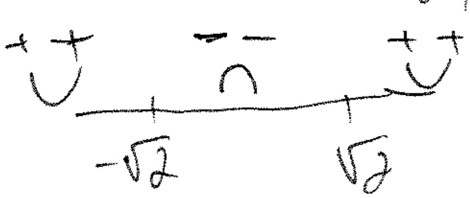
$$= \frac{-6x}{(x^2-2)^2}$$



rel. max

$$y'' = \frac{(x^2-2)^2(-6) - (-6x)2(x^2-2)(2x)}{(x^2-2)^2} =$$

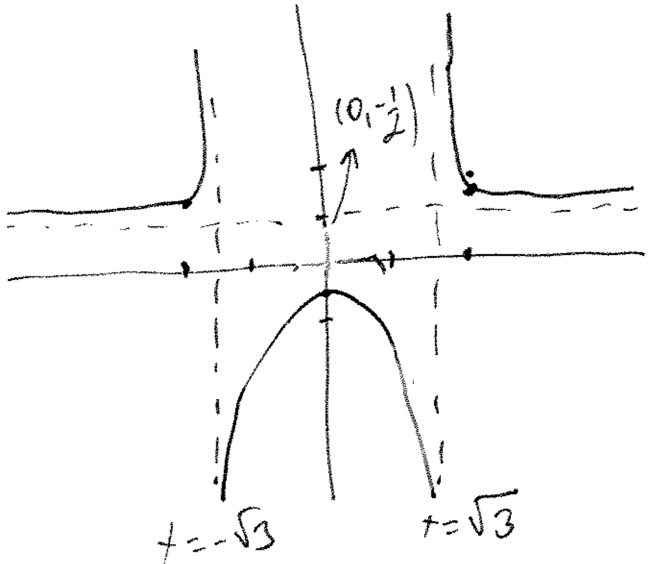
$$= \frac{(x^2-2)(-6x^2+12+24x^2)}{(x^2-2)^4} = \frac{18x^2+12}{(x^2-2)^3}$$



$f(2) =$

$= \frac{4+1}{4-2} = \frac{5}{3}$

$f(-2) = \frac{5}{3}$



c) $y = x\sqrt{4-x^2}$

Domain $4-x^2 \geq 0 \Rightarrow x \in [-2, +2]$

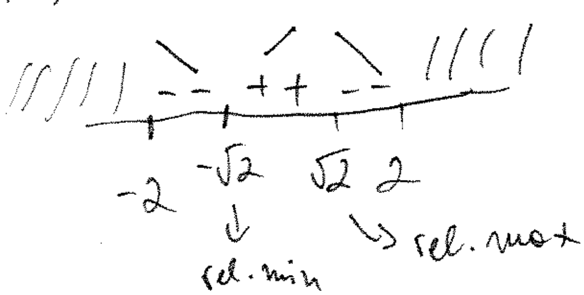
y-intercept 0

x intercept : 0, ± 2

$y' = \sqrt{4-x^2} + \frac{x}{2}\frac{-2x}{\sqrt{4-x^2}} = \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} = 0$

$\frac{4-x^2-x^2}{\sqrt{4-x^2}} = 0$

$\Rightarrow \frac{4-\cancel{2}x^2}{\sqrt{4-x^2}} = \frac{2(2-x^2)}{\sqrt{4-x^2}} \Rightarrow$ critical point $x = \pm 2$, $x = \pm\sqrt{2}$



$1.323 - \frac{2.25}{1.323}$

$f(-\sqrt{2}) = -\sqrt{2}\sqrt{2} = -2$

$f(\sqrt{2}) = 2$

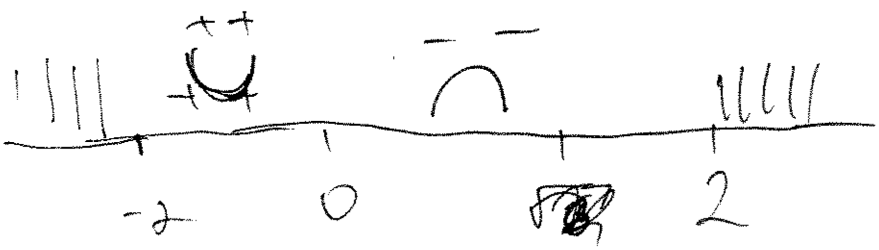
$$y'' = \frac{2\sqrt{4-x^2}\,(-2x) - (2-x^2)\dfrac{1}{2\sqrt{4-x^2}}(-2x)}{(4-x^2)}$$

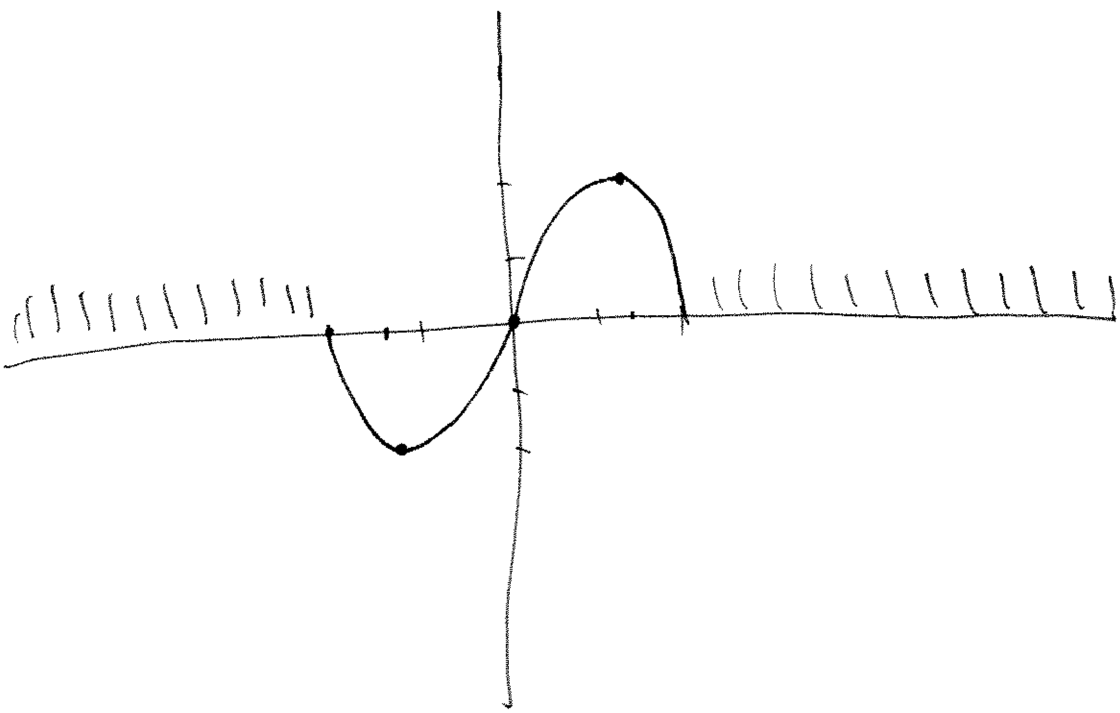
$$= \cancel{2}\ \cancel{\frac{-4x\sqrt{4-x^2}}{}}$$

$$= \frac{-4x(4-x^2) + 4x - 2x^3}{(4-x^2)^{3/2}} = \frac{-16x+4x^3+4x-2x^3}{(4-x^2)^{3/2}}$$

$$= \frac{2x^3-12x}{(4-x^2)^{3/2}} = \frac{2x(x^2-6)}{(4-x^2)^{3/2}}$$

$x = \pm\sqrt{6}$ is not in our domain





III d)

$$y = x + \frac{32}{x^2} = \frac{x^3+32}{x^2}$$

H. A. $\underline{\underline{\text{No}}}$

V. A. $x = 0 \Rightarrow$ No y-intercept

x-intercept $\frac{x^3+32}{x^2} = 0 \Rightarrow x^3 + 32 = 0$

$$\Rightarrow x^3 = -32 \Rightarrow x = -\sqrt[3]{32} = -2\sqrt[3]{4}$$

$$y' = 1 - \frac{2(32)}{x^3} = 1 - \frac{64}{x^3}$$

$$1 - \frac{64}{x^3} = 0 \Rightarrow x^3 = 64 \Rightarrow x = 4$$

+++   ---   +++

$\underbrace{0 \qquad\qquad 4}_{\text{crit. points}}$

$x = 0$ is a V. A (from above)

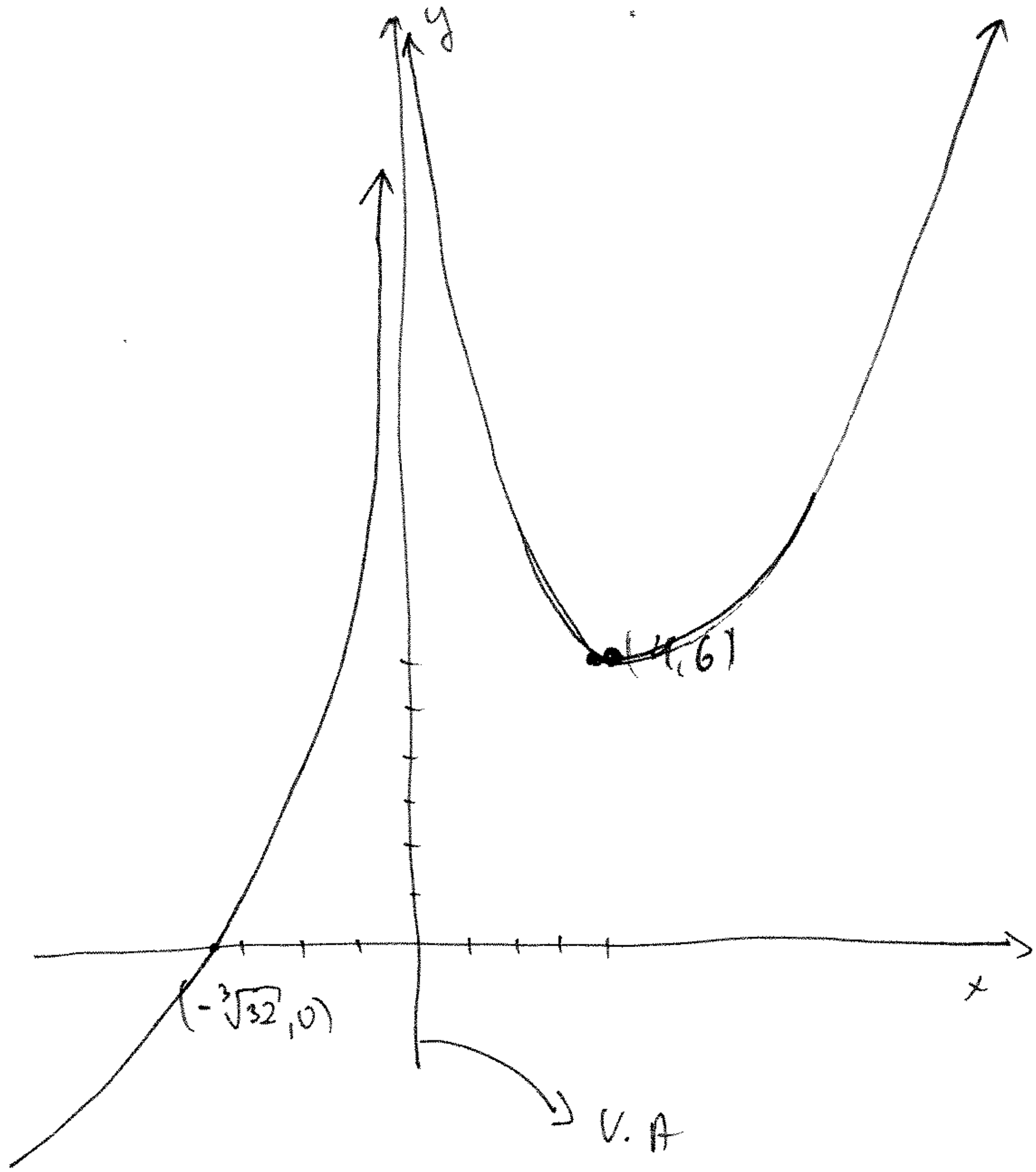
$x = 4$ is rel. min

$$f(4) = 4 + \frac{32}{16} = 6$$

$$y'' = \frac{3(64)}{x^4}$$

++   ++

0



y
x
(4, 6)
$(-\sqrt[3]{32}, 0)$
V. A

e) $y = \dfrac{x^4}{x^4-1} = \dfrac{x^4}{(x^2-1)(x^2+1)} = \dfrac{x^4}{(x-1)(x+1)(x^2+1)}$

H. A. $y = 1$ y-intercept : $y = 0 \Rightarrow (0,0)$

V. A. $x = \pm 1$ x-intercept $\dfrac{x^4}{x^4-1} = 0 \Rightarrow x = 0$

$$y' = \frac{(x^4-1)(4x^3) - x^4(4x^3)}{(x^4-1)^2} = \frac{\cancel{4x^7} - 4x^3 - \cancel{4x^7}}{(x^4-1)^2}$$

crit. points: $x = 0, \pm 1$

+ + + + + + + − − − −

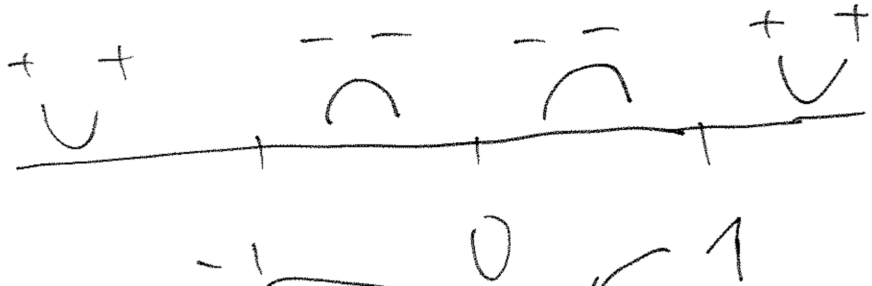
−1 0 1

0 → rel. max

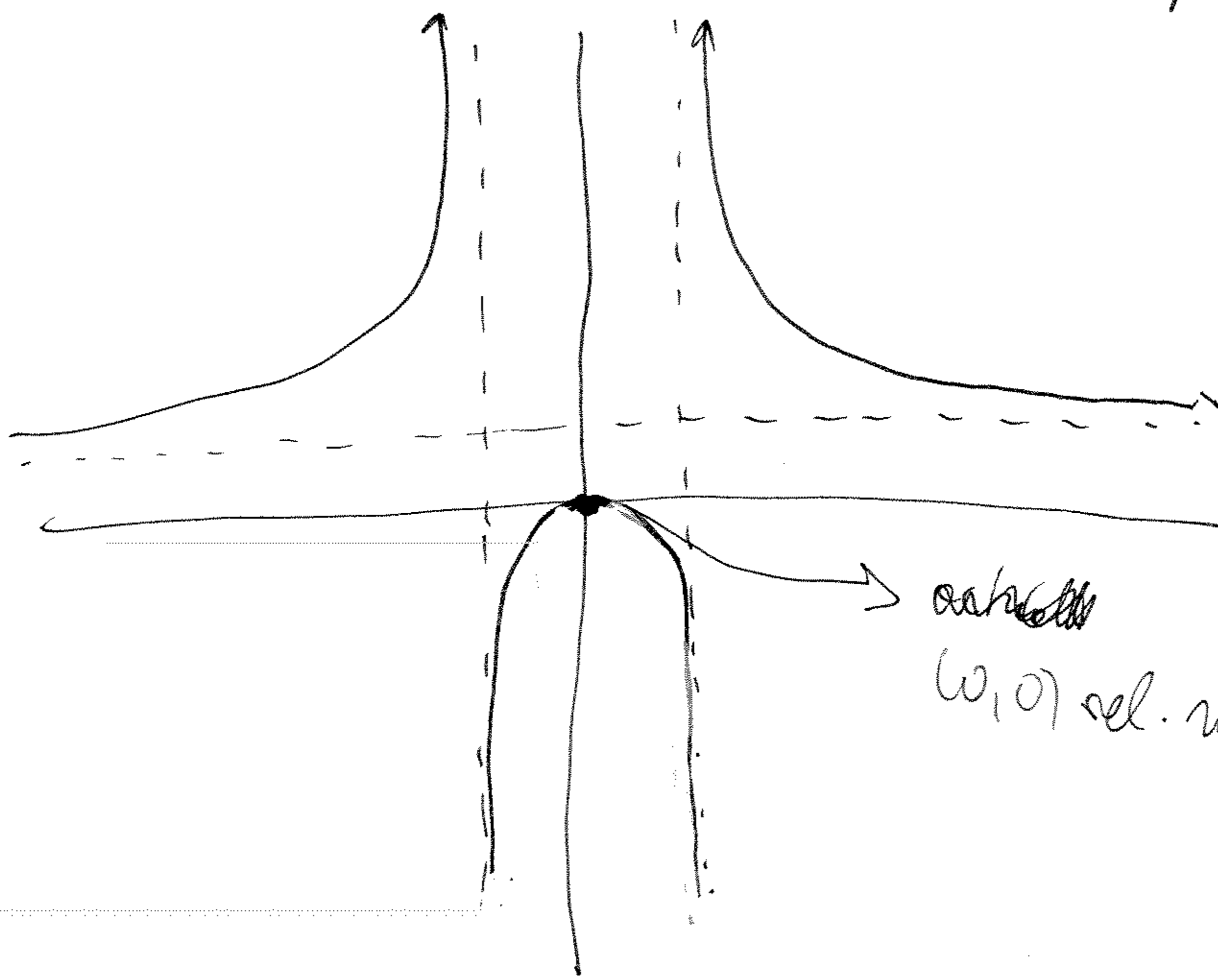
$$y'' = \frac{(x^4-1)^2(-12x^2) - (-4x^3)\,2(x^4-1)(4x^3)}{(x^4-1)^4}$$

$$= \frac{\cancel{(x^4-1)}\left(-12x^2(x^4-1) + 32x^6\right)}{(x^4-1)^{\cancel{4}\,3}} = \frac{20x^6+12x^2}{(x^4-1)^3}$$

$$= \frac{x^2(20x^4+12)}{(x^4-1)^3}$$



not on inflection point



(0,0) rel. max.

f) $f(x) = \dfrac{x^2-9}{x^2-4x+3} = \dfrac{(x-3)(x+3)}{(x-1)(x-3)}$

H.A: $y = 1$

V.A. Try $x = 1, 3$

Plug in $x = 1$ $\quad f(1) = \dfrac{-8}{0} \Rightarrow x = 1$ is a V.A

Plug in $x = 3$ $\quad f(3) = \dfrac{0}{0} \Rightarrow \lim\limits_{x \to 3} \dfrac{x^2-9}{x^2-4x+3} = \lim\limits_{x \to 3} \dfrac{\cancel{(x-3)}(x+3)}{(x-1)\cancel{(x-3)}}$

$= \dfrac{6}{2} = 3$

Domain: $(-\infty, 1) \cup (1, 3) \cup (3, +\infty)$

y-intercept: -3

x-intercept: $x = -3$ ($x = 3$ is not since $f(x)$ is not defined there)

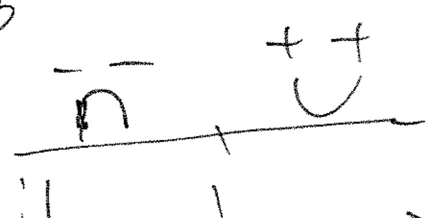
$$f'(x) = \frac{(x^2-4x+3)(2x) - (x^2-9)(2x-4)}{(x^2-4x+3)^2}$$

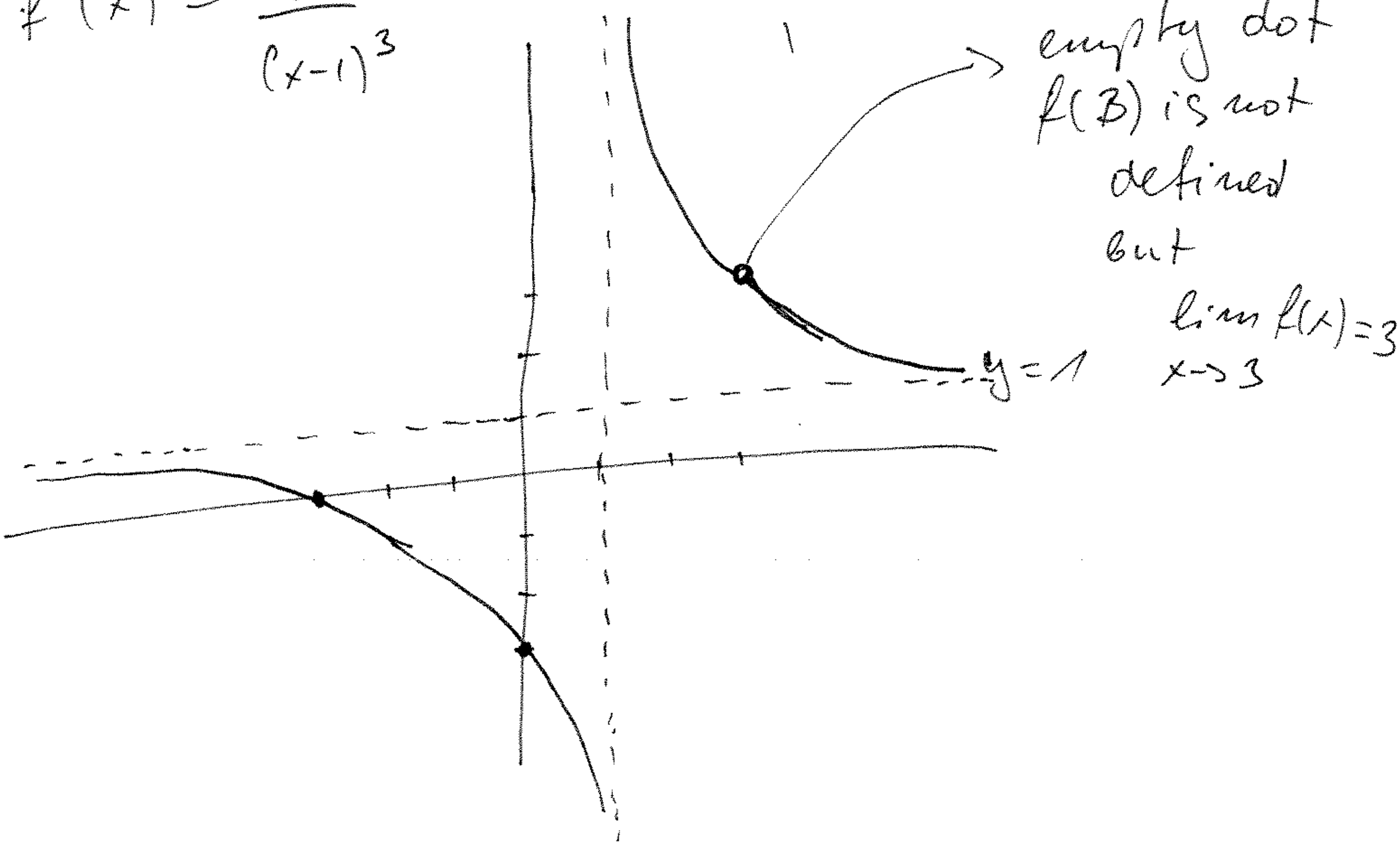
$$= \frac{\cancel{2x^3} - 8x^2 + 6x - \cancel{2x^3} + 4x^2 + 18x - 36}{(x-1)^2(x-3)^2}$$

$$= \frac{-4x^2 + 24x - 36}{(x-1)^2(x-3)^2} = \frac{-4(x^2 - 6x + 9)}{(x-1)^2(x-3)^2} =$$

$$= \frac{-4\cancel{(x-3)^2}}{(x-1)^2\cancel{(x-3)^2}} = -\frac{4}{(x-1)^2} \Rightarrow \text{No rel. max.}$$

$$f''(x) = \frac{8}{(x-1)^3}$$





VII b) $f(x) = e^{1/x} = e^{-x}$

$f'(x) = e^{-x}(-1) = -e^{-x}$

c) $e^{\sqrt{x}} = e^{x^{1/2}}$

$g'(x) = e^{x^{1/2}}\left(\frac{1}{2}x^{-3/2}\right)$

$= \frac{e^{\sqrt{x}}}{2\sqrt{x^3}}$

e) $f'(x) = \frac{4}{2}(e^x+e^{-x})^3(e^x+e^{-x}(-1))$

$= 2(e^x+e^{-x})^3(e^x-e^{-x})$

f) $y' = 2\cancel{x}e^x + x^2e^x - \cancel{2e^x} - 2\cancel{x}\cancel{e^x} + \cancel{2e^x} = x^2e^x$

d) $y' = 12x^2e^{-x} - 4x^3e^{-x}$

VIII a) $2xy + x^2\frac{dy}{dx} - e^x - xe^x = 0$

$\frac{dy}{dx} = \frac{xe^x + e^x - 2xy}{x^2}$

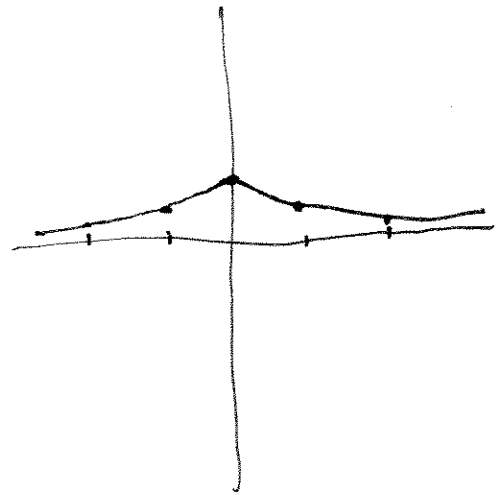
b) $e^{xy}(y) + e^{xy}\left(x\frac{dy}{dx}\right) + 2x - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x - ye^{xy}}{(xe^{xy} - 2y)}$

IV c) $y = 2^{-x^2} = \frac{1}{2^{x^2}}$ $\lim_{x\to\infty} \frac{1}{2^{x^2}} = 0$

$y = 0$ is a H.A.

x	-2	-1	0	1	2
$-x^2$	-4	-1	0	-1	-4
2^{-x^2}	$\frac{1}{16}$ ≈ 0.0625	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{16}$



$\cancel{y = \frac{1}{2^{x^2}} \quad \lim_{x\to\infty} \frac{1}{2^{x^2}} = \frac{1}{2^{x^2}} =}$

V a) $5^{x+1} = 125$

$5^{x+1} = 5^3$

$\Rightarrow x+1 = 3$

$x = 2$

b) $\left(\frac{1}{5}\right)^{2x} = 625$

$5^{-2x} = 5^4$

$-2x = 4$

$x = -2$

c) $4^2 = (x+2)^2$

$\Rightarrow 4 = x+2$

$\Rightarrow x = 2$

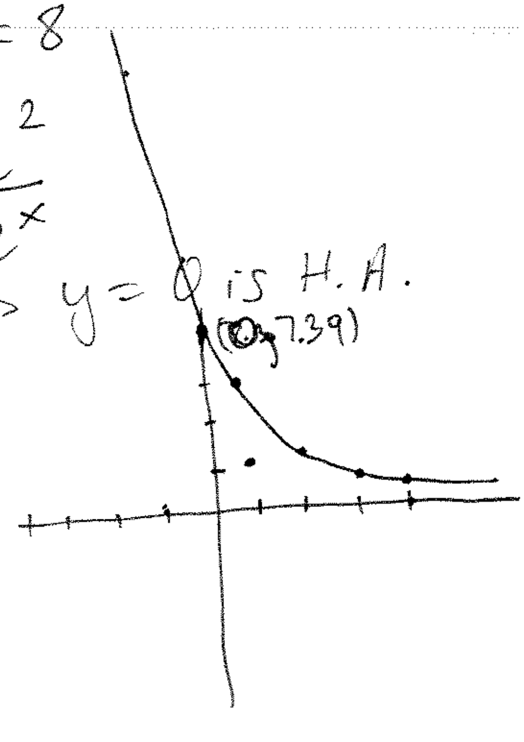
d) $5^2 = (x-3)^2$

$\Rightarrow 5 = x-3$

$x = 8$

VI $g(x) = e^{-x+2} = e^2 e^{-x} = \frac{e^2}{e^x}$

$\lim_{x\to\infty} e^{-x+2} = \lim_{x\to\infty} \frac{e^2}{e^x} = 0 \rightarrow y = 0$ is H.A.



x	4	3	2	1	0	-1
$-x+2$	-2	-1	0	$+1$	2	3
e^{-x+2}	e^{-2} ≈ 0.135	e^{-1} ≈ 0.368	1	2.718 $= e^1$	e^2 ≈ 7.39	e^3 ≈ 20

1.X a) $x+y=100 \Rightarrow y=100-x$

$P=xy \Rightarrow P=x(100-x)=100x-x^2$

$P'=100-2x$ Prove it is a max

$100-2x=0$

$x=50$

$\Rightarrow y=50$

$+ \; + \quad - \; -$

50

b) $x+2y=100 \Rightarrow y=\frac{100-x}{2}$

$P=xy \Rightarrow P=\frac{1}{2}(100x-x^2)$

$P'=\frac{1}{2}(100-2x)=50-x$

$+ \; + \quad - \; -$

$\Rightarrow x=50$

50

$\Rightarrow y=25$

c) See answers in the book (the back of the book)

d) Back of the book

e) $x-y=50 \Rightarrow y=x-50$

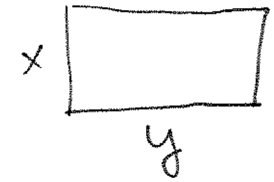
$P=xy \Rightarrow P=x(x-50)=x^2-50x$

$P'=2x-50 \Rightarrow x=25$

$- \; - \quad + \; +$

25

X $P = 2x + 2y = 200$



$\Rightarrow x + y = 100 \Rightarrow y = 100 - x$

$A = xy = x(100 - x) = 100x - x^2$

Prove it is a max

$+++ \quad --- $
50

$A' = 100 - 2x$

$100 - 2x = 0 \Rightarrow x = 50\,m. \Rightarrow y = 50\,m$

XI $A = xy = 100 \Rightarrow y = \frac{100}{x}$

$P = 2x + 2y = 2x + \frac{200}{x}$

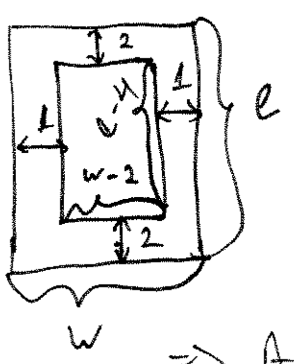
$P' = 2 - \frac{200}{x^2}$ $\qquad 2 - \frac{200}{x^2} = 0 \Rightarrow x = \pm 10$

$-- \quad ++$
$-10 \quad 0 \quad 10$

crit. points

$\Rightarrow \boxed{x = 10 \text{ is a min}}$

XII $(w-2)(\ell-4) = 30$



want to minimize $A = \ell w$

$\Rightarrow \ell - 4 = \frac{30}{w-2}$

$\Rightarrow \ell = \frac{30}{w-2} + 4$

$\Rightarrow A = \left(\frac{30}{w-2} + 4\right) w = \frac{30w}{w-2} + 4w$

$A' = \frac{(w-2)30 - 30w}{(w-2)^2} + 4 = -\frac{60}{(w-2)^2} + 4 = \frac{-60 + 4w^2 - 16w +}{(w-2)^2}$

$$A' = \frac{4w^2 - 16w - 44}{(w-2)^2} = \frac{4(w^2 - 4w - 11)}{(w-2)^2}$$

$$w^2 - 4w - 11 = 0$$

$$w_{1,2} = \frac{4 \pm \sqrt{16+44}}{2} = \frac{4 \pm 2\sqrt{15}}{2} = \begin{cases} 2+\sqrt{15} \\ 2-\sqrt{15} \end{cases}$$

$$\begin{array}{ccccc} & & - - - & & + + + \\ 2-\sqrt{15} & & 2 & & 2+\sqrt{15} \end{array}$$

$\Rightarrow$ $w = 2+\sqrt{15} \approx 5.87$ is a min

$$l = \frac{30}{3.87} + 4 = 11.75$$

13. See book (answer at the back)
14. See quiz (solutions on the web)
15. See book (answ. at the back)
20. $V = x^2 y = 83\frac{1}{3} = \frac{250}{3} \Rightarrow y = \frac{250}{3x^2}$

Minimize $S = 3xy + x^2 = 3x \dfrac{250}{3x^2} + x^2$

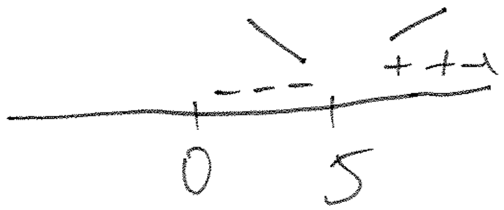
$$= \frac{250}{x} + x^2$$

$$S' = -\frac{250}{x^2} + 2x \qquad \frac{-250}{x^2} + 2x = 0$$

$$-250 + 2x^3 = 0$$

$$x^3 = 125 \Rightarrow x = 5$$

0) prove $x=5$ is a min



$\Rightarrow$ $x=5$ is a min.

$$\Rightarrow y = \frac{250}{3(5)^2} = \frac{250}{75} = \frac{10}{3}$$

#16 See book (answer at the back)

#17 $V = \underset{l\ w}{(6-2x)^2}\ \underset{h}{x}\ \cancel{+h}$ $\leftarrow$ Maximize

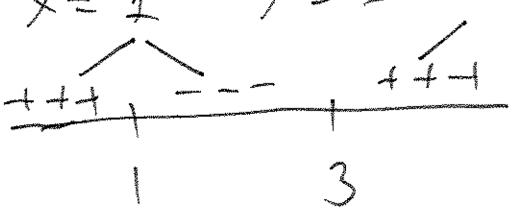
$$V' = 2(6-2x)(-2)x + (6-2x)^2$$

$$= -24x + 8x^2 + 36 - 24x + 4x^2$$

$$= 12x^2 - 48x + 36 = 12(x^2 - 4x + 3)$$

$$\cancel{12}(x-1)(x-3) = 0$$

$\Rightarrow$ $x=1$ $\quad$ $x=3$



$\Rightarrow$ $x=1$ is a max

$\Rightarrow V = 16\ in^3$

18 a) $\bar{C} = \frac{C}{x} = 0.001x^2 + 5 + \frac{250}{x}$

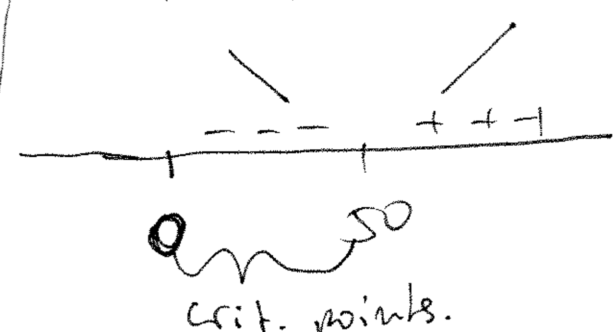
$\bar{C}' = 0.002x - \frac{250}{x^2}$

$0.002x - \frac{250}{x^2} = 0$

$0.002x = \frac{250}{x^2}$

$\rightarrow 0.002x^3 = 250 \rightarrow x^3 = \frac{250}{0.002}$

$x = 50$



19) See back of the book for answer

20) Do it yourself

\# 33 $x = p^2 - 20p + 100$

$p = \$2 \qquad x = 2^2 - 20(2) + 100 = 104 - 40 = 64$

$p_n = 2 + 0.05(2) = \$2.1$

$\Rightarrow x_n = (2.1)^2 - 20(2.1) + 100 =$

$= 4.41 - 42 + 100 = 62.41$

percent change in quantity $= \dfrac{62.41 - 64}{64} = -2.48\%$ → decrease in units sold

b) Average elasticity $= \dfrac{-2.48\%}{+5\%} = -0.496$ - inelastic

c) Implicit differentiation

$1 = 2p\dfrac{dp}{dx} - 20\dfrac{dp}{dx}$

$\dfrac{dp}{dx} = \dfrac{1}{2p-20} \qquad \left.\dfrac{dp}{dx}\right|_{p=9} = \dfrac{1}{18-20} = -\dfrac{1}{2}$

$\eta = \dfrac{p/x}{dp/dx} = \dfrac{9}{-\frac{1}{2}} = -18 \Rightarrow$ elastic (much more than when $p = \$2$)

$x = 81 - 180 + 100 = 1$

I think the book meant to use $p = \$2$. Then you will get the answer from the back!

(d) $R = xp = (p^2 - 20p + 100)p = p^3 - 20p^2 + 100p$

$R' = 3p^2 - 40p + 100$

$3p^2 - 40p + 100 = 0$

$$p_{1,2} = \frac{40 \pm \sqrt{1600 - 1200}}{6} = \frac{40 \pm 20}{6} \begin{cases} 10 \\ \frac{20}{6} = \frac{10}{3} = \$3.33 \end{cases}$$

$+ + \quad \frac{10}{3} \quad - - - \quad 10 \quad + + +$ $\Rightarrow$ max @ $p = \$\frac{10}{3} = \$3.33$

$$x = \frac{100}{9} - \frac{200}{3} + 100 = \frac{100 - 600 + 900}{9} = \frac{400}{9} \text{ units}$$

#34 $\quad p^3 + x^3 = 9$

a) Implicit differentiation:

$$3p^2 \frac{dp}{dx} + 3x^2 = 0$$

$$\frac{dp}{dx} = \frac{-3x^2}{3p^2} = -\frac{x^2}{p^2} \qquad \left.\frac{dp}{dx}\right|_{x=2} = \frac{-4}{1} = -4$$

$$8 + p^3 = 9 \Rightarrow p^3 = 1 \text{ or } x = 1$$

$$|\eta| = \left| \frac{1/2}{-4} \right| = \frac{1}{8} \quad \text{Inelastic}$$

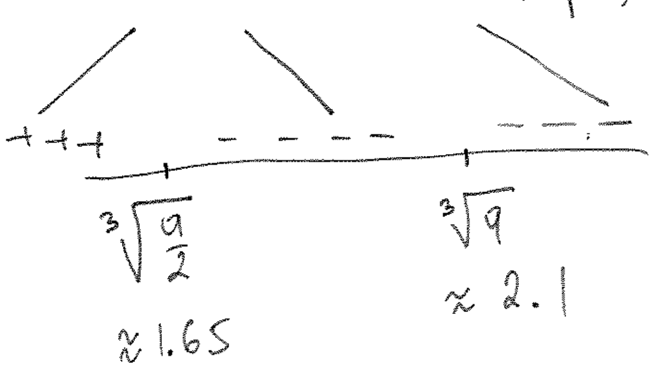
b) $R = xp$

$$p^3 + x^3 = 9 \Rightarrow x = \sqrt[3]{9 - p^3}$$

$$\Rightarrow R = p(9 - p^3)^{1/3}$$

$$R' = (9 - p^3)^{1/3} + \frac{p}{\cancel{3}}(9 - p^3)^{-2/3}(-\cancel{3}p^2)$$

$$= (9 - p^3)^{1/3} - \frac{p^3}{(9 - p^3)^{2/3}} = \frac{9 - p^3 - p^3}{(9 - p^3)^{2/3}} = \frac{9 - 2p^3}{(9 - p^3)^{2/3}}$$



$$\Rightarrow \text{max @ } p = \sqrt[3]{\frac{9}{2}} \approx 1.65$$

$$x = \sqrt[3]{9 - \frac{9}{2}} = \sqrt[3]{4.5}$$

c) $x = \sqrt[3]{\frac{9}{2}}$

$p = \sqrt[3]{\frac{9}{2}}$

$$\frac{dp}{dx} = -\frac{x^2}{p^2} = -\left(\frac{x}{p}\right)^2 = -\left(\frac{\sqrt[3]{\frac{9}{2}}}{\sqrt[3]{\frac{9}{2}}}\right)^2 = -1$$

$$|\eta| = \left|\frac{p/x}{dp/dx}\right| = \left|\frac{\sqrt[3]{9/2} / \sqrt[3]{9/2}}{-1}\right| = |-1| = 1$$

40.) c) $r_{eff} = \left(1+\frac{r}{n}\right)^n - 1 = \left(1+\frac{0.075}{4}\right)^4 - 1 = 7.71\%$

42) $A = 21,154.03$

$$A = P\left(1+\frac{r}{n}\right)^{nt} \Rightarrow 21,154.03 = P\left(1+\frac{0.078}{12}\right)^{4(12)}$$

$\Rightarrow 21,154.03 = 1.365\ P$

$\Rightarrow P \approx 15497.46$

44. $A = 6,000\left(1+\frac{0.0625}{12}\right)^{12(3)} = 7233.86$