Midterm Review Answs. and solutions

I] a) $y = \frac{3x^2-4x}{6x}$

$$y' = \frac{6x(6x-4)-(3x^2-4x)6}{36x^2} =$$

$$= \frac{36x^2 - \cancel{24x} - 18x^2 + \cancel{24x}}{36x^2} = \frac{18x^2}{36x^2} = \frac{1}{2}$$

Easier way: $y = \frac{3x^2-4x}{6x} = \frac{\cancel{x}(3x-4)}{6\cancel{x}} = \frac{3x-4}{6}$

$$= \frac{3x}{6} - \frac{4}{6} = \frac{1}{2}x - \frac{2}{3}$$

$\Rightarrow y' = \frac{1}{2}$

b) $y = \frac{x^2-4}{x+2} = \frac{(x-2)\cancel{(x+2)}}{\cancel{x+2}} = x-2$

$\Rightarrow y' = 1$

c) $f(x) = (x^5-1)(4x^2-7x-3)$

$$f'(x) = (x^5-1)(8x-7) + (4x^2-7x-3)(5x^4)$$

$$= 8x^6 - 7x^5 - 8x + 7 + 20x^6 - 35x^5 - 15x^4$$

$$= 26x^6 - 42x^5 - 15x^4 - 8x + 7$$

d) $f(x) = \sqrt[3]{x}\,(x+1) = x^{1/3}(x+1)$

$f'(x) = \frac{1}{3}x^{-2/3}(x+1) + x^{1/3}(1) =$

$= \frac{1}{3}x^{-2/3+1} + \frac{1}{3}x^{-2/3} + x^{1/3} =$

$= \frac{1}{3}x^{1/3} + \frac{1}{3}x^{-2/3} + x^{1/3} = \frac{4}{3}x^{1/3} + \frac{1}{3}x^{-2/3}$

e) $h'(t) = \frac{(t^2+5t+6)(1) - (t+2)(2t+5)}{(t^2+5t+6)^2} =$

$= \frac{t^2+\cancel{5t}+6-2t^2-4t-\cancel{5t}-10}{(t^2+5t+6)^2} = \frac{-t^2-4t-4}{(t^2+5t+6)^2}$

$= \frac{-(t+2)^2}{(t+2)^2(t+3)^2} = -\frac{1}{(t+3)^2}$

f) $f(x) = \frac{x+1}{x^{1/2}}$

Two ways: use quotient rule or:

$f(x) = \frac{x}{x^{1/2}} + \frac{1}{x^{1/2}} = x^{1/2} + x^{-1/2}$

$f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}$

g) $f(x) = (x^5-3x)\left(\frac{1}{x^2}\right) = x^3 - \frac{3}{x} = x^3 - 3x^{-1}$

$f'(x) = 3x^2 + \frac{3}{x^2}$

h) $f(x) = \dfrac{x^3+3x+2}{x^2-1}$

$$f'(x) = \frac{(x^2-1)(3x^2+3) - (x^3+3x+2)(2x)}{(x^2-1)^2}$$

$$= \frac{3x^4 + \cancel{3x^2} - \cancel{3x^2} - 3 - 2x^4 - 6x^2 - 4x}{(x^2-1)^2}$$

$$= \frac{x^4 - 6x^2 - 4x - 3}{(x^2-1)^2}$$

II.

a) $s(t) = (t^2+3t-1)^{-1}$

$s'(t) = -(t^2+3t-1)^{-2}(2t+3) = -\dfrac{2t+3}{(t^2+3t-1)^2}$

b) $y = \dfrac{1}{\sqrt{x+2}} = (x+2)^{-1/2}$

$y' = -\frac{1}{2}(x+2)^{-3/2}(1)$

c) $g(x) = 3(x^3-1)^{-1/3}$

$g'(x) = -1(x^3-1)^{-4/3}(3x^2)$

d) $f'(x) = 3(4x-x^2)^2(4-2x)$

II e) $y = x^{1/2}(x-2)^7$

$$y' = x^{1/2}\, 7(x-2)^6 + \frac{1}{2}x^{-1/2}(x-2)^7$$

f) $g(t) = \dfrac{3t^2}{(t^2+2t-1)^{1/2}}$

$$g'(t) = \frac{(t^2+2t-1)^{1/2}(6t) - \frac{3t^2}{2}(t^2+2t-1)^{-1/2}(2t+2)}{t^2+2t-1}$$

$$= \frac{6t\sqrt{t^2+2t-1} - 3t^2(t+1)(t^2+2t-1)^{-1/2}}{t^2+2t-1}$$

If you want you can simplify further, but this is enough

$$g'(t) = \frac{6t}{\sqrt{t^2+2t-1}} - \frac{3t^2(t+1)}{\sqrt{(t^2+2t-1)^3}}$$

g) $y' = 3\left(\dfrac{4x^2}{3-x}\right)^2\left(\dfrac{(3-x)8x - 4x^2(-1)}{(3-x)^2}\right)$

$$= 3\,\frac{16x^4}{(3-x)^2}\,\frac{24x-8x^2+4x^2}{(3-x)^2} =$$

$$= 3 \cdot \frac{48x^4(24x-4x^2)}{(3-x)^4}$$

h) $f'(x) = 3x^2(x-4)^5 + 5x^3(x-4)^4$

i) $y = (x+2)^{-1/2}$ $\quad y' = -\frac{1}{2}(x+2)^{-3/2}$

III a) $g(t) = -4(t+2)^{-2}$

$g'(t) = 8(t+2)^{-3}$

$g''(t) = -24(t+2)^{-4} = -\frac{24}{(t+2)^4}$

b) $f(x) = x^4 - 2x^3$

$f' = 4x^3 - 6x^2$

$f'' = 12x^2 - 12x$

$f''' = 24x - 12$

c) $g(x) = x^{-1}$

$g'(x) = -x^{-2}$

$g''(x) = 2x^{-3}$

$g'''(x) = -6x^{-4}$

$g^{IV}(x) = 24x^{-5} = \frac{24}{x5}$

IV a) $x - \frac{dy}{dx} = 6$

$\frac{dy}{dx} = x - 6$

b) $4x^2y - 3y^{-1} = 0$

$8xy + 4x^2\frac{dy}{dx} + \frac{3}{y^2}\frac{dy}{dx} = 0 \;/y^2$

$8xy^3 + 4x^2y^2\frac{dy}{dx} + 3\frac{dy}{dx} = 0$

b) $\frac{dy}{dx} = \frac{-8xy^3}{4x^2y^2+3}$

c) $xy^2+4xy=10$

$y^2+2xy\frac{dy}{dx}+4y+4x\frac{dy}{dx}=0$

$\frac{dy}{dx} = -\frac{y^2+4y}{2xy+4x}$

d) $\frac{xy-y^2}{y-x}=1 \Rightarrow \frac{-y(\cancel{y-x})}{\cancel{y-x}}=1$

$\Rightarrow -y=1 \Rightarrow -\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=0$

e) $\frac{(x-5y)\left(2+\frac{dy}{dx}\right)-(2x+y)\left(1-5\frac{dy}{dx}\right)}{(x-5y)^2}=0 \quad \Big/\times(x-5$

$\Rightarrow (x-5y)\left(2+\frac{dy}{dx}\right)-(2x+y)\left(1-5\frac{dy}{dx}\right)=0$

$\Rightarrow \cancel{2x}-10y+(x-5y)\frac{dy}{dx}-\cancel{2x}-y+5(2x+y)\frac{dy}{dx}=0$

$$(x - \cancel{5y} + 10x + \cancel{5y})\frac{dy}{dx} = 11y$$

$$11x\frac{dy}{dx} = 11y$$

$$\boxed{\frac{dy}{dx} = \frac{y}{x}}$$

You can solve it other ways:

$$\cancel{(x-5y)}\frac{2x+y}{\cancel{x-5y}} = 1(x-5y)$$

$$2x+y = x-5y$$

$$\Rightarrow 6y = -x$$

$$y = -\frac{x}{6}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{6}$$

This may seem different from above answ, but if you substitute $y = -\frac{x}{6}$ you'll get the same using

$$\frac{dy}{dx} = \frac{y}{x} = \frac{-\frac{x}{6}}{x} = -\frac{x}{6}\cdot\frac{1}{x} = -\frac{1}{6}$$

V

a) $x^2+y^2=9$

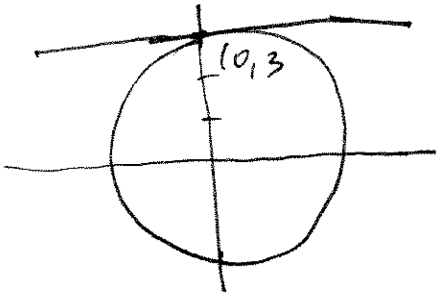
$2x+2y\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}=\frac{-2x}{2y}=-\frac{x}{y}\Big|_{(0,3)}=-\frac{0}{3}=0$

$\Rightarrow m=0$

$y=mx+b \Rightarrow 3=0(0)+b \Rightarrow$

$\Rightarrow b=3 \Rightarrow y=3$



b) $4xy+x^2=5$

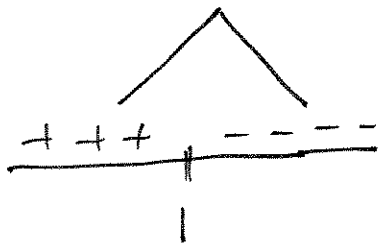
$4x\frac{dy}{dx}+4y+2x=0$

$\frac{dy}{dx}=\frac{-4y-2x}{4x}=\frac{-2y-x}{2x}\Big|_{(1,1)}$

$\Rightarrow m=\frac{-2(1)-(1)}{2(1)}=\frac{-2-1}{2}=-\frac{3}{2}$

$y=mx+b \Rightarrow y=-\frac{3}{2}x+b \Rightarrow 1=-\frac{3}{2}+b \Rightarrow b=\frac{5}{2}$

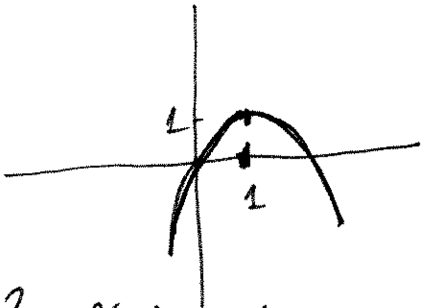
$y=-\frac{3}{2}x+\frac{5}{2}$



VI a) $y' = -2x+2$

$-2x+2=0$

$x = 1$ → critical pnt.



⇒ increasing $(-\infty,1)$

decreasing $(1,\infty)$

⇒ rel. max @ $x=1$ $y=-(1)^2+2(1)=1$

b) $f(x)=\sqrt{4-x^2} = (4-x^2)^{1/2}$ Domain $4-x^2 \geq 0$

$f'(x)=\frac{1}{2}(4-x^2)^{-1/2}(-2x) = -\frac{2x}{\sqrt{4-x^2}}$

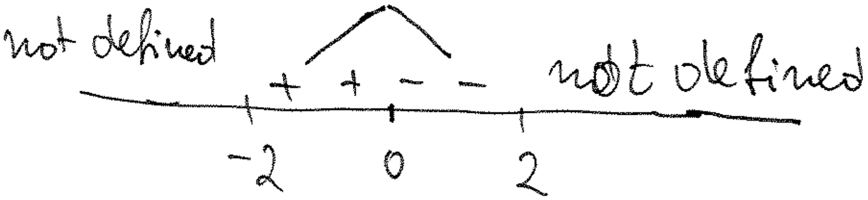
critical pts: where f' is undefined ⇒

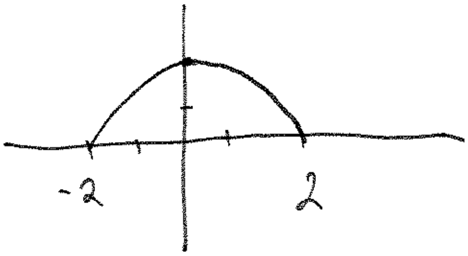
⇒ $x=\pm 2$ and $f'=0$ ⇒ $x=0$

Domain of $f(x)$: $4-x^2 \geq 0$ $(2-x)(2+x) \geq 0$

Domain $x \in [-2,2]$

⇒ Thus only 2 intervals to test





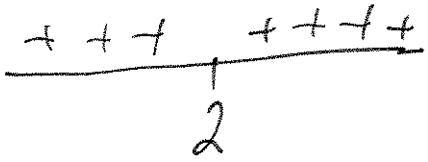
⇒ decreasing on $(0,2]$

increasing on $(-2,0)$

actually 0 is rel. max

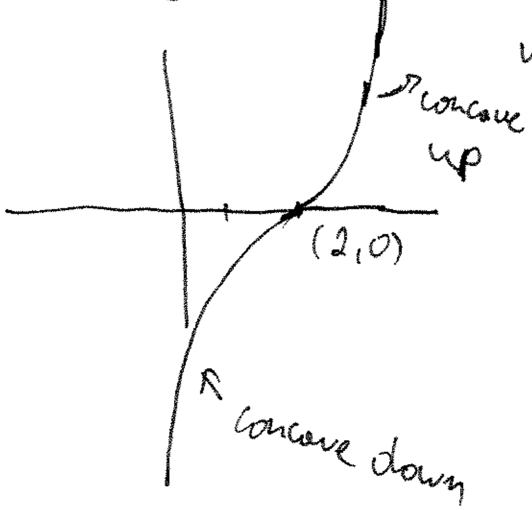
c) $y = (x-2)^3$

$y' = 3(x-2)^2$



=> critical pnt $x = 2$

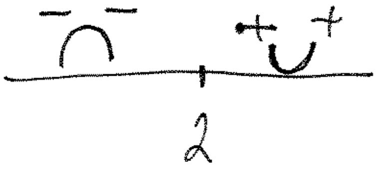
=> y is increasing $(-\infty, \infty)$



when $x = 2$ $y = 0$

you can also check for concavity to help you draw the graph

$y'' = 6(x-2)$



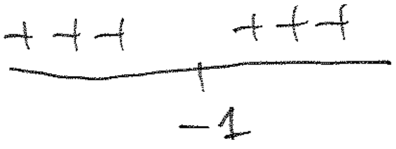
d) $f(x) = \frac{x}{x+1}$

$$f'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}$$

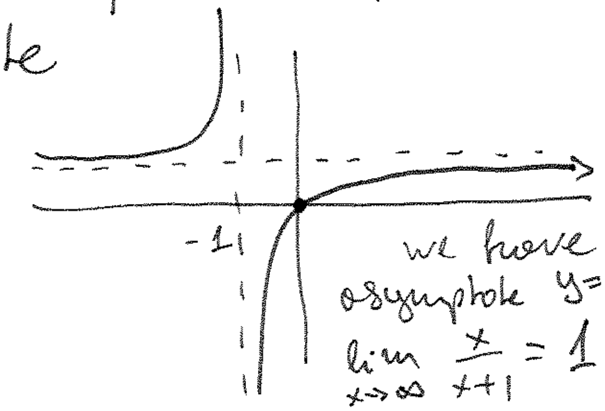
crit. pnt. $x = -1$ (where f' is undefined)

=> But also $x = -1$ is wher f is undefined =>

=> $x = -1$ is an asymptote



increasing $(-\infty, \infty)$



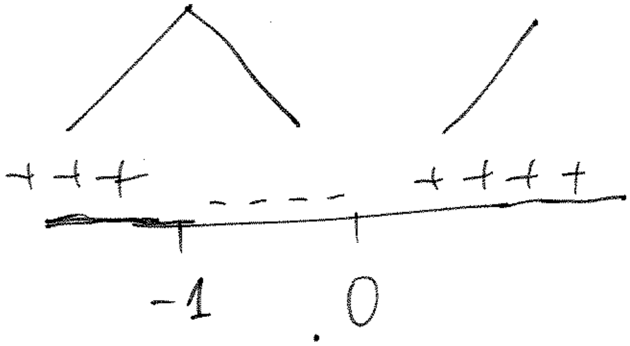
we have a horizont asymptote $y = 1$

$\lim_{x \to \infty} \frac{x}{x+1} = 1$ $\lim_{x \to -\infty} \frac{x}{x+1} = 1$

e) $y = \begin{cases} 2x+1 & x \leq -1 \\ x^2-2 & x > -1 \end{cases}$

The func. is continuous $2(-1)+1 = -1 = (-1)^2-2$

$y' = \begin{cases} 2 & x < -1 \\ 2x & x > -1 \end{cases}$



$y' = 0$ when $x = 0$

y' is not continuous:

$2 \neq -2$

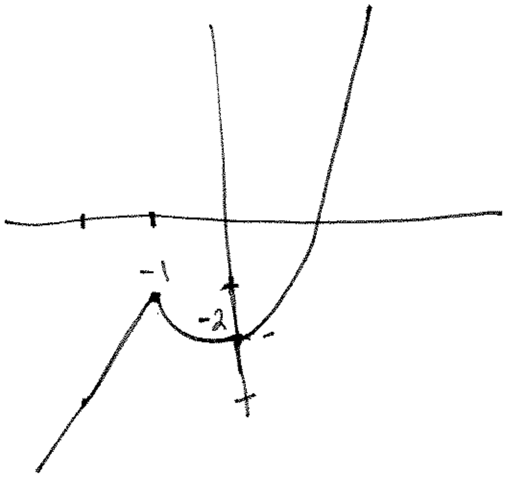
=> critical pts 0, -1

increasing $(-\infty,-1) \cup (0,+\infty)$

decreasing $(-1,0)$

=> $x=0$ rel. min

$x = -1$ rel. max



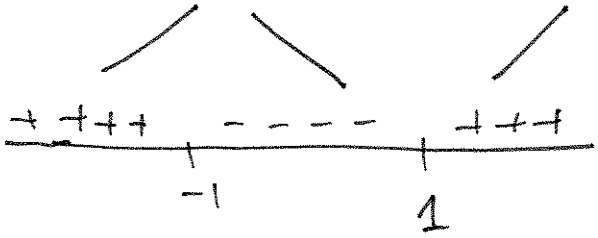
VII a) $g'(x) = x^4 - 1 = (x^2-1)(x^2+1) =$

$= (x-1)(x+1)(x^2+1)$

$\Rightarrow (x-1)(x+1)(x^2+1) = 0$

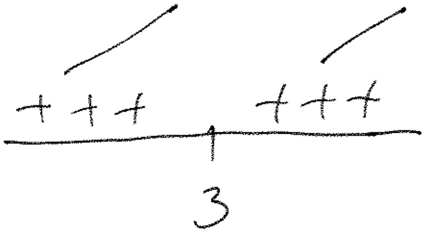
$\Rightarrow x = 1 \quad x = -1$ → critical pts ⇒ $x = -1$ rel. max.

$x = 1$ rel min.



b) $h'(x) = 6(x-3)^2$

$x = 3$ critical pnt.



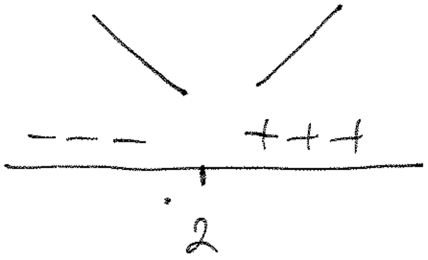
⇒ ~~max~~ no rel. max, no rel. min

c) $f'(x) = 4x^3 - 32 = 4(x^3-8) = 4(x-2)(x^2+2x+4)$

⇒ $x=2$ is critical pnt.

$x^2 + 2x + 4 = 0$

$x_{1,2} = \dfrac{-2 \pm \sqrt{4-16}}{2}$ ⇒ no solution



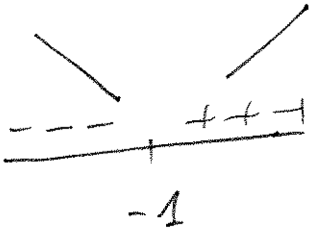
⇒ $x=2$ is rel. min

VII d) $f'(x) = 4x^3 - 36x^2 = 4x^2(x-9)$

$\Rightarrow x = 0,\ x = 9$ critical pnts.

$\Rightarrow$ rel. min @ $x = 9$
no rel. max.

VIII a) $f'(x) = 2x + 2$



$\Rightarrow 2x + 2 = 0 \Rightarrow x = -1$

$\Rightarrow$ rel. min @ $x = -1$

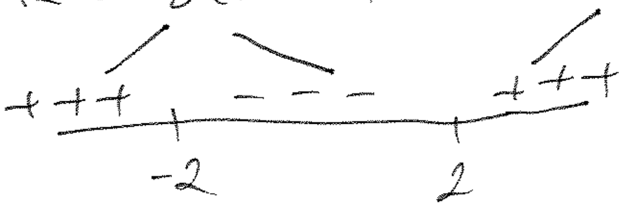
$f(-1) = (-1)^2 + 2(-1) - 4 = 1 - 2 - 4 = -5$

$f(1) = 1 + 2 - 4 = -1$

$\Rightarrow$ abs. max $(1,-1)$
abs. min $(-1,-5)$

b) $f'(x) = 3x^2 - 12 = 3(x^2-4) = 3(x-2)(x+2)$

$x = \pm 2$



$\Rightarrow$ rel. max @ $x = -2$
rel. min @ $x = 2$

$\Rightarrow$ abs. max $(4,16)$
abs min $(2,-16)$

$f(0) = 0$

$f(2) = 2^3 - 12(2) = 8 - 24 = -16$

$f(4) = 64 - 48 = 16$

c) $h(t) = \dfrac{t}{t-2}$

$$h'(t) = \frac{(t-2)-t}{(t-2)^2} = \frac{t-2-t}{(t-2)^2} = -\frac{2}{(t-2)^2}$$

critical point @ $t=2$, but there is an asymptote there since $h(t)$ is not defined @ $t=2$

$$\left.\begin{aligned} h(3) &= \frac{3}{3-2} = 3 \\ h(5) &= \frac{5}{5-2} = \frac{5}{3} \end{aligned}\right\} \Rightarrow \begin{aligned} &\text{abs. min } (5, \tfrac{5}{3}) \\ &\text{abs. max } (3,3) \end{aligned}$$

d) $f(x) = \dfrac{8}{x+1} = 8(x+1)^{-1}$

$$f'(x) = -8(x+1)^{-2} = \frac{-8}{(x+1)^{+2}}$$

$\Rightarrow$ critical point @ $x=-1$, but also $x=-1$ is asymptote.

$$\left.\begin{aligned} f(0) &= \frac{8}{1} = 8 \\ \lim_{x\to\infty} \frac{8}{x+1} &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &\text{rel min } x=\infty, \ y \neq 0 \\ &\text{rel. max } (0,8) \end{aligned}$$

IX Note: We have not studied IInd derivative test, yet. So use 1st derivative test

a) $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$

$\Rightarrow x=0$, $x=3$ critical pnts.

$\Rightarrow$ rel. min @ $x=3$

b) $f'(x) = 1 - \frac{4}{x^2}$

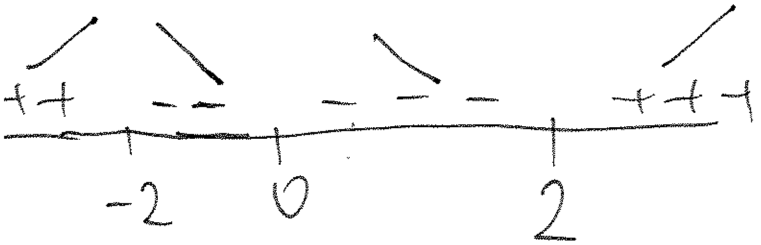
$1 - \frac{4}{x^2} = 0 \;/ \cdot x^2$

$x^2 - 4 = 0$

$(x-2)(x+2) = 0 \Rightarrow x = \pm 2$

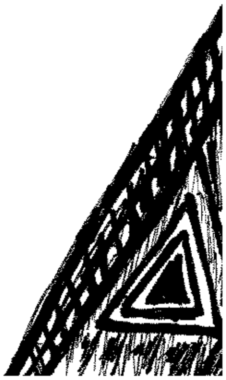
Also $x=0$ makes $f'(x)$ undefined,

but also it makes $f(x)$ undefined $\Rightarrow$ asymptote @ $x=0$



$\Rightarrow$ rel. max @ $x=-2$

rel. min @ $x=2$



c) $f'(x) = \frac{(x^2-1) - x(2x)}{(x^2-1)^2} = \frac{x^2-1-2x^2}{(x^2-1)^2} =$

$= \frac{-x^2-1}{(x^2-1)^2} = \frac{-(x^2+1)}{(x^2-1)^2}$

critical points @ $x = \pm 1$ (these make f' undefined)

But $x = \pm 1$ make $f(x)$ undefined => asymptotes

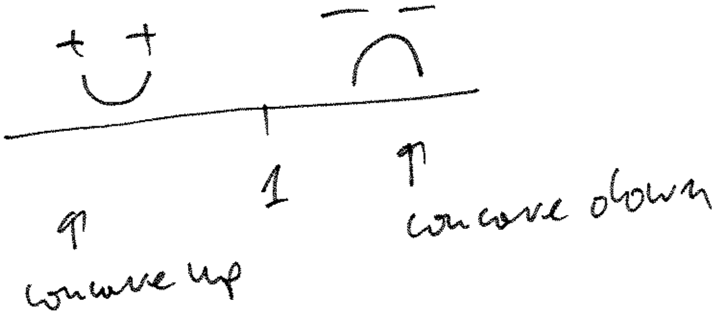
No rel. min or max.

X a) $y = -x^3 + 3x^2 - 2$

$y' = -3x^2 + 6x$

$y'' = -6x + 6 = -6(x-1)$

=> $x = 1$



b) $f'(x) = \frac{(4-x^2)(2x) - (x^2+4)(-2x)}{(4-x^2)^2}$

$= \frac{8x - \cancel{2x^3} + \cancel{2x^3} + 8x}{(4-x^2)^2} = \frac{16x}{(4-x^2)^2}$

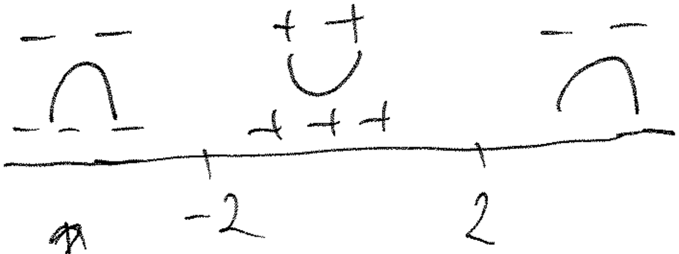
$$f''(x) = \frac{(4-x^2)^2(16) - 16x(2)(4-x^2)(-2x)}{(4-x^2)^4}$$

$$= \frac{16(4-x^2)(4-x^2+64x^2)}{(4-x^2)^4} =$$

$$= \frac{16(4+63x^2)}{(4-x^2)^3}$$

$f''(x) \neq 0$ for all x

f'' is undefined @ $x = \pm 2$



c) $$f'(x) = \frac{(x^2+1)(2x) - x^2(2x)}{(x^2+1)^2} =$$

$$= \frac{3x^2+2x-3x^2}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}$$

$$f''(x) = \frac{2(x^2+1)^2 - 4x(x^2+1)(2x)}{(x^2+1)^4} =$$

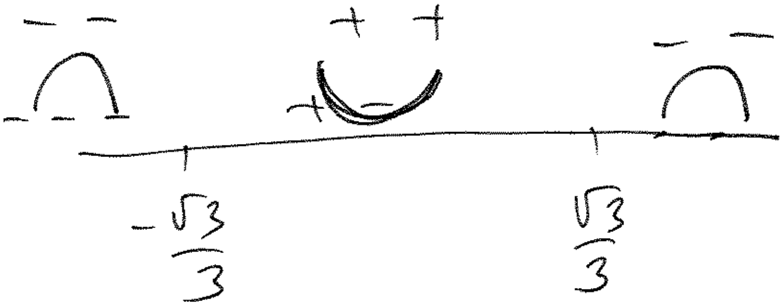
$$= \frac{2(x^2+1)(x^2+1-4x^2)}{(x^2+1)^4} = \frac{2(1-3x^2)}{(x^2+1)^3}$$

$\Rightarrow f''(x) = 0$

$\rightarrow 1 - 3x^2 = 0$

$\Rightarrow x^2 = \frac{1}{3}$

$x = \pm\sqrt{\frac{1}{3}} = \pm \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \pm \frac{\sqrt{3}}{3}$



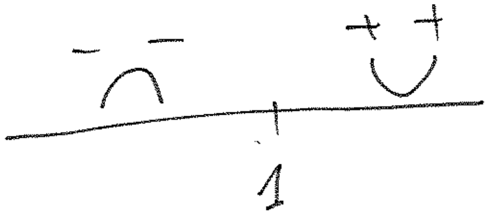
d) $y' = 5x^4 + 20x^3 - 80x$

$y'' = 20x^3 + 60x^2 - 80 = 20(x^3 + 3x - 4)$

$= 20(x^3 + \underline{4x} - x \underline{-4}) = 20(x(x^2-1) + 4(x-1))$

$= 20(x(x-1)(x+1) + 4(x-1)) = 20(x-1)(x^2+x+4)$

$\rightarrow x = 1$



XI e)

func is increasing => $f'(x)$ is positive

func. is concave down => f'' is negative

3) func is decreasing => f' is negative

func is concave up => f'' is positive

XII. a) $C(9) = 3.6\sqrt{9} + 500 = 510.8$

$C(10) = 511.38$

$C(10) - C(9) = 511.38 - 510.8 = 0.58^{\$}$

b) $C = 3.6\,x^{1/2} + 500$

$C'(x) = \frac{3.6}{2}\,x^{-1/2} = \frac{1.8}{\sqrt{x}}$

$C'(9) = \frac{1.8}{3} = 0.6^{\$}/\text{unit}$

XIII a) $R(14) = 2(14)(900 + 32(14) - 14^2)$

$= 28(900 + 448 - 196) = {}^{\$}32{,}256$

$R(15) = 30(900 + 480 - 225) = 34{,}650$

$R(15) - R(14) = {}^{\$}2\,394$

b) $R' = 2(900 + 32x - x^2) + 2x(32 - 2x) =$

$= 1800 + 64x - 2x^2 + 64x - 4x^2 =$

$= 1800 + 128x - 6x^2$

$R'(14) = {}^{\$}2\,416$

XIV

a) $P(150) = -1125 + 3000 - 1000 = 875$

$P(151) = -1140.05 + 3020 - 1000 = 879.95$

$P(151) - P(150) = 4.95$

b) $P'(x) = -0.1x + 20$

$P'(150) = -15 + 20 = 5$