SPATIAL BOUNDS ON THE EFFECTIVE COMPLEX PERMITTIVITY FOR TIME-HARMONIC WAVES IN RANDOM MEDIA

DAVID C. DOBSON AND LYUBIMA B. SIMEONOVA *

Abstract. When we consider wave propagation in one-, two-, and three-dimensional random medium in the case when the wave length is finite, scattering effects must be accounted for and the effective dielectric coefficient is no longer a constant, but a spatially dependent function. We present an upper bound on the effective permittivity and a bound on its spatial variations that depends on the maximum volume of the inhomogeneities and the contrast of the medium.

Key words. Random media, effective properties

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1. Background.

Usually, when one considers the propagation of an electromagnetic wave in a random medium, two length scales are of importance. The first scale is the wavelength λ of the electromagnetic wave probing the medium. The second one is the typical scale of the inhomogeneities δ. There has been plenty of work to build an effective medium theory that is applicable to wave propagation and fields that oscillate with time provided that the wavelengths associated with the fields are much larger than the microstructure. This limit where the size of the microstructure goes to zero is called the quasistatic or infinite wavelength limit. In this case the heterogeneous material is replaced by a homogeneous fictitious one whose macroscopic characteristics are good approximations of the initial ones. The solutions of a boundary value partial differential equation describing the propagation of waves converge to the solution of a limit boundary value problem which is explicitly described when the size of the heterogeneities goes to zero.

The problem of finding bounds on the effective properties of materials in the quasistatic limit has been investigated vigorously, and there have been significant advances not only in deriving optimal bounds, but also in describing the materials that accomplish these bounds [14] and references within. Wellander and Kristensson [19] and Conca and Vanninathan [3] have both recently analyzed the homogenization of time-harmonic wave problems in periodic media, using entirely different methods. Their results are each applicable to problems in which the wavelength of the incident field is much larger than the microstructure.

For waves in random media, Keller and Karal [12] and Papanicolaou [16] use averaging of random realizations of materials in order to describe the effective properties of the composites when interacting with electromagnetic waves. Both analyses assume that the random materials deviate slightly from a homogeneous material, i.e. the contrast of the random inclusions is small. Keller and Karal assume *a priori* that the effective dielectric coefficient is a constant. Using perturbation methods, they approximate the dielectric constant with a complex number, whose imaginary part accounts for the wave attenuation.

A great overview of the subject of wave propagation in random media is given in a book by Ishimaru [11]. Also recent results in this field could be found in the AMS-IMS-SIAM proceedings edited by Kuchment [13].

*Both authors: Department of Mathematics, University of Utah, Salt Lake City, UT 84112-0090, USA

The above methods that provide bounds and describe the behavior of the dielectric coefficients do not account for scattering effects which occur when the wavelength is no longer much larger than the inhomogeneities of the composite and when the contrast is large. This problem has remained open and results are sparse. The problem is difficult and none of the techniques that come from the quasistatic regime can be applied directly to the scattering problem since all of the quasistatic methods utilize the condition that the size of the heterogeneities goes to zero.

Even the correct definition of "effective medium" is somewhat unclear outside the quasistatic regime. In this work, we assume that the purpose of the effective medium is to reproduce the average or expected wave field as the actual medium varies over a given set of random realizations. In the quasistatic case, the effective permittivity ε^* is defined by

$$\varepsilon^* \langle E \rangle = \langle D \rangle = \langle \varepsilon E \rangle,$$

where the averaged electric field $\langle E \rangle = \bar{E}$ is a given constant, and the averaged dielectric displacement $\langle D \rangle$ is independent of x which ensures that ε^* in the quasistatic case is a constant.

For simplicity in this work we consider waves in two- or three-dimensional random media governed by the Helmholtz equation

$$\triangle u + \omega^2 \varepsilon u = 0,$$

where the permittivity function $\varepsilon(x)$ assumes random realizations within some probability space. We average over all the possible material realizations to obtain the equation

$$\triangle \langle u \rangle + \omega^2 \langle \varepsilon u \rangle = f,$$

where $\langle \rangle$ denotes expected value, i.e. averaging over the set of realizations, and not a spatial average. We seek to find the dielectric coefficient ε^* that will solve the problem

$$\triangle \langle u \rangle + \omega^2 \varepsilon^* \langle u \rangle = f, \tag{1.1}$$

where $\langle u \rangle$ is the averaged solution. From the above two equations, it is easy to see that the appropriate definition for ε^* is

$$\varepsilon^* = \frac{\langle \varepsilon u \rangle}{\langle u \rangle}. \tag{1.2}$$

Note that the definition of ε^* does not preclude spatial variations $\varepsilon^* = \varepsilon^*(x)$.

Wave localization and cancellation must be accounted for when the wavelength is in the same order as the size of the heterogeneities, which means that the effective coefficients are no longer necessarily constants as in the quasistatic case, but functions of the space variable. We have illustrated in a previous paper [18] that as ω increases (which will decrease the wavelength), we begin to see spatial variations in the effective dielectric coefficient due to the presence of scattering effects. Nevertheless ε^* as defined in (1.2) is a "correct" definition of the effective dielectric coefficient, in that it reproduces the average field response through equation (1.1).

Since ε^* cannot be calculated explicitly in general, to be useful in applications it is important that we can bound both ε^* itself, and some measure of the spatial variations in ε^*. The main result of this paper, presented in Theorem 3.1, is a bound on the magnitude of ε^* and a bound on the total variation, $\|\varepsilon^*\|_{BV}$. The estimates hold for any fixed frequency $\omega > 0$ and show an explicit dependence on the feature size and contrast of the random medium.

The paper is organized as follows. We pose the model problem of electromagnetic wave propagation in a composite material in subsection 2.1. The two-component composite material is random and its structure is described using random variables to describe its geometry and component dependence in subsection 2.2. Existence and uniqueness of solutions and uniform bound on the solutions are obtained and Lipschitz bounds of the solutions with respect to the dielectric coefficients of the materials are derived in subsection 2.3. Both the uniform bound on the solutions and the Lipschitz bounds are instrumental in the results' derivations and proofs. Spatial variations due to scattering effects are allowed. Bounds on the effective dielectric coefficient and its spatial variations are obtained when certain conditions are satisfied. These results are stated in the theorem in section 3, which is proven using novel methods that incorporate both analytical concepts from the theory of partial differential equations and probability arguments.

2. Model Problem.

2.1. Electromagnetic wave propagation. Consider time-harmonic electromagnetic wave propagation through nonmagnetic ($\mu = 1$) heterogeneous media. Assuming that the electric field vector $E = (0, 0, u)$ and ε is independent of x_3, Maxwell's equations reduce to the Helmholtz equation

$$\triangle u + \omega^2 \varepsilon u = 0, \tag{2.1}$$

where ω represents the frequency, and $\varepsilon \in L^\infty(\mathbb{R}^n)$ is the dielectric coefficient. In three dimensions, each field component satisfies (2.1).

Let our bounded spatial domain be $\Omega \subseteq \mathbb{R}^n$, where $n = 2, 3$. The region outside Ω is filled with a homogeneous material. In particular, assume for $x \notin \Omega$, we have $\varepsilon(x) = 1$. Call S_0 the sphere of radius R_0, i.e. $S_0 = \{r = R_0\}$, and let $\Omega_0 = \{|x| < R_0\}$.

Outside the ball Ω_0, we separate the solution u to (2.1) into the incident and scattered field: $u = u_i + u_s$. The scattered field u_s can also be separated. Wellposedness of the problem requires to imposing Sommerfeld's radiation condition as boundary condition at infinity, i.e.

$$\lim_{r \to \infty} r^{\frac{n-1}{2}} \left(\frac{\partial}{\partial r} - i\omega \right) u = 0,$$

uniformly in all directions, where $n = 2, 3$ is the spatial dimension. Here, it is assumed that the time-harmonic field is $e^{-i\omega t} u$.

Let the linear operator $T\colon H^{\frac{1}{2}}(S_0) \to H^{-\frac{1}{2}}(S_0)$ (Dirichlet-to-Neumann map) define the relationship between the traces $u_s|_{\{r=R_0\}}$ and $\partial_r u_s|_{\{r=R_0\}}$: $T(u_s|_{\{r=R_0\}}) = (\partial_r u_s)|_{\{r=R_0\}}$. The Dirichlet-to-Neumann operator defines an exact nonreflecting condition on the artificial boundary S_0, i.e. there are no spurious reflections of the scattered solution introduced at S_0. We write T explicitly for the two- and three-dimensional cases in the Appendix. On the boundary $S_0 = \{r = R_0\}$, the solution $u = u_i + u_s$ should then satisfy

$$\partial_r u - Tu = \partial_r u_i - Tu_i + \partial_r u_s - Tu_s = \partial_r u_i - Tu_i = c.$$

In this way the problem on $\mathbb{R}^n$ is equivalently replaced by

$$\begin{aligned} \triangle u + \omega^2 \varepsilon u = 0 \quad & \text{in } \Omega_0 \supset \Omega, \\ \left(\frac{\partial u}{\partial r} - Tu \right) = c \quad & \text{on } S_0. \end{aligned}$$

2.2. Random structure. We are interested in computing expected values of wave fields as the underlying medium ranges over some class of random materials. In this section, we define the probability space characterizing these materials.

We fill our bounded domain by random cell materials; see e.g. Milton [14] Section 15. Our two-phase random materials are constructed as follows. The first step is to divide the bounded domain into a finite number of cells. The cells vary in size and topology, but their volume and perimeter are bounded by a parameter δ.

The second step is to randomly assign each cell as material of phase ϵ_0 with probability p or ϵ_1 with probability $1-p$ in a way that is uncorrelated both with the shape of the cell and with the phases assigned to the surrounding cells.

We have the probability space $(\Psi_\delta, \mathcal{J}_\delta, P_\delta)$, where Ψ_δ is a set of material realizations with a σ-algebra $\mathcal{J}_\delta$ of subsets of Ψ_δ, and a probability measure P_δ on $\mathcal{J}_\delta$ with $P_\delta(\Psi_\delta) = 1$.

Elements $\psi \in \Psi_\delta$ are characterized by two random variables, $\psi = (m, g)$, where the variable m depends on the random variable g. The variable g describes the geometry of the material by partitioning the domain Ω in N_g parts, each of which is filled either with material ε_0 or material ε_1, which is done by the random variable m. Thus, g describes the subdivision of our domain into subdomains and once the geometry g is fixed, the random variable m distributes the material in the subdomains. The variable $g \in \Gamma_\delta$, partitions the spatial domain Ω into N_g disjoint subdomains $\{\Omega_j\}_{j=1}^{N_g}$ such that $\cup \Omega_j = \Omega$. Here Γ_δ is some set of partitions of Ω. And the variable $m_g = \{m_1, \dots, m_{N_g}\}$ assigns zero for material ε_0 with probability p or one for material ε_1 with probability $1-p$ in each spatial subdomain. Thus, the real part of the dielectric constant in the composite material is:

$$\varepsilon_{m,g}(x) = \begin{cases} \varepsilon_0 & \text{if } m_j = 0 \text{ and } x \in \Omega_j; \\ \varepsilon_1 & \text{if } m_j = 1 \text{ and } x \in \Omega_j. \end{cases}$$

We assume without loss of generality that $\varepsilon_1 > \varepsilon_0$.

$(\Psi_\delta, \mathcal{J}_\delta, P_\delta)$ depends on a parameter $\delta > 0$, that bounds the volume of each subdomain $\{\Omega_j\}_{j=1}^{N_g}$: $|\Omega_j| \leq \delta$ and the perimeter of each subdomain $|\partial \Omega_j| \leq \delta$. Since the volume of Ω is fixed, as δ decreases, the set of realizations Ψ_δ changes.

Once the spatial domain Ω is partitioned into N_g disjoint subdomains $\{\Omega_j\}_{j=1}^{N_g}$, the material in each subdomain is assigned. This gives us one realization. The set of realizations Ψ_δ is infinite in general since the set of all geometries, Γ_δ, can be infinite. Fix a geometry g. Denote the set of realizations for geometry g by R_g:

$$R_g = \{m_g = (m_1, \dots, m_{N_g}) : m_j = 0 \text{ or } m_j = 1,\, j = 1, \dots, N_g\}$$

The set R_g has 2^{N_g} elements. Thus the set of material realizations, Ψ_δ is described as follows,

$$\Psi_\delta = \{(g, m_g) : g \in \Gamma_\delta,\, m_g \in R_g\}.$$

The probability measure is

$$P = \sum_{m_g \in R_g} \prod_{j=1}^{N_g} p^{1-m_j} \left(1-p\right)^{m_j} G_\delta,$$

where G_δ is the probability measure on the space of all geometries, Γ_δ. The product describes the multiplication of the probabilities of the materials in each subdomain Ω_j, which is summed over the set of all realizations for a particular geometry g.

2.3. Existence and uniqueness of solutions and Lipschitz bounds. For a fixed dissipation constant $\epsilon_i > 0$, define a set

$$\mathcal{A} := \{\varepsilon = \varepsilon_r + i\,\varepsilon_i : \varepsilon_r = \varepsilon_{m,g} \text{ for some } (m,g) \in \Psi_\delta\}.$$

Given an incident field u_i, we must solve the following problem

$$\triangle u + \omega^2 \varepsilon_r u + i\omega^2 \varepsilon_i u = 0 \quad \text{in } \Omega_0 \tag{2.2}$$

$$\left(\frac{\partial u}{\partial r} - Tu\right) = c \quad \text{on } S_0. \tag{2.3}$$

Existence and uniqueness of weak solutions, with a uniform bound, may be obtained for materials with a little bit of absorption, i.e. $\varepsilon_i > 0$.

Throughout the remainder of the paper, in order to simplify estimates within proofs, C will denote a constant which is independent of (ε, u), whose value may change from line to line.

LEMMA 2.1. *For each $\varepsilon \in \mathcal{A}$ with $\varepsilon_i > 0$, problem (2.2)-(2.3) admits a unique weak solution $u \in H^2(\Omega)$. Furthermore, there exists a constant C depending on ε_i, $\mathcal{A}$, such that $\|u\|_{H^2(\Omega)} \leq C$, independent of $\varepsilon \in \mathcal{A}$.*

Proof. The ideas for the proof of the lemma come from the proof of a similar lemma in [5]. Define for $u, v \in H^1(\Omega)$

$$a(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v} - \omega^2 \int_\Omega \varepsilon u \overline{v} - \int_{S_0} (Tu)\overline{v},$$

and

$$b(v) = c \int_{S_0} \overline{v}.$$

Using bounds (5.3) and (5.6) for the two- and three-dimensional problem respectively, it is straightforward to show that $a(u,v)$ defines a bounded sesquilinear form over $H^1(\Omega) \times H^1(\Omega)$, and that $b(v)$ is a bounded linear functional on $H^1(\Omega)$. Weak solutions $u \in H^1(\Omega)$ of (2.2) solve the variational problem

$$a(u,v) = b(v) \quad \text{for all } v \in H^1(\Omega). \tag{2.4}$$

The sesquilinear form a uniquely defines a linear operator $A : H^1(\Omega) \to H^1(\Omega)$ such that $a(u,v) = \langle Au, v\rangle_{H^1(\Omega)}$, and the functional $b(v)$ is uniquely identified with an element $b \in H^1(\Omega)$ such that $b(v) = \langle b, v\rangle$. By reflexivity and an abuse of notation Problem (2.4) is then equivalently stated

$$Au = b. \tag{2.5}$$

We intend to show that a is coercive by establishing a bound $|a(u,u)| \geq c > 0$ for all $u \in H^1(\Omega)$ with $\|u\|_{H^1(\Omega)} = 1$. We have

$$a(u,u) = \int_\Omega |\nabla u|^2 - \omega^2 \int_\Omega \varepsilon_r |u|^2 - Re\left(\int_{S_0} (Tu)\overline{u}\right)$$
$$-i\, Im\left(\int_{S_0} (Tu)\overline{u}\right) - i\omega^2 \varepsilon_i \int_\Omega |u|^2.$$

For the two-dimensional problem we have

$$\int_{S_0} (Tu)\overline{u} = \int_{S_0} \sum_{m=1}^{\infty} \gamma_m \hat{u}_m e^{im\theta} \overline{u} = \sum_{m=1}^{\infty} \gamma_m |\hat{u}_m|^2,$$

where $\hat{u}_m$ are the Fourier coefficients of u (See Appendix). Since $Re(\gamma_m) < 0$ and $Im(\gamma_m) > 0$ for every m,

$$Re\left(\int_{S_0} (Tu)\overline{u}\right) < 0 \quad \text{and} \quad Im\left(\int_{S_0} (Tu)\overline{u}\right) > 0.$$

Similarly, for the three-dimensional case

$$\int_{S_0} (Tu)\overline{u} = \int_{S_0} \sum_{l=0}^{\infty} \gamma_l \sum_{m=-l}^{l} \hat{u}_{lm} Y_{lm} \overline{u} = \sum_{l=0}^{\infty} \gamma_l \sum_{m=-l}^{l} |\hat{u}_{lm}|^2,$$

where $\hat{u}_{lm}$ are the coefficients in the spherical harmonics expansion of u (See Appendix). Since $Re(\gamma_l) < 0$ and $Im(\gamma_l) > 0$ for every l ,

$$Re\left(\int_{S_0} (Tu)\overline{u}\right) < 0 \quad \text{and} \quad Im\left(\int_{S_0} (Tu)\overline{u}\right) > 0.$$

Assuming $\|u\|^2_{H^1(\Omega)} = \int_\Omega |\nabla u|^2 + \int_\Omega |u|^2 = 1$, and noticing that the first four terms on the right-hand side are purely real and the last two terms are purely imaginary, we find

$$2|a(u,u)| \geq \left|1 - \int_\Omega (1 + \omega^2 \varepsilon_r |u|^2 - Re\left(\int_{S_0} (Tu)\overline{u}\right)\right|$$
$$+ \left|-\omega^2 \varepsilon_i \int_\Omega |u|^2 - Im\left(\int_{S_0} (Tu)\overline{u}\right)\right|.$$

For convenience, write $r = \int_\Omega (1 + \omega^2 \varepsilon_r)|u|^2$, $s = \int_\Omega |u|^2$, and

$$t = \begin{cases} -\sum_{m=1}^{\infty} Re(\gamma_m)|\hat{u}_m|^2 & \text{in two dimensions;} \\ -\sum_{l=0}^{\infty} Re(\gamma_l) \sum_{m=-l}^{l} |\hat{u}_{lm}|^2 & \text{in three dimensions.} \end{cases}$$

Obviously t, r, and s are nonnegative real numbers which depend on u (and ε in the case of r). Although t and s are essentially independent, r must satisfy

$$(1 + \omega^2 \varepsilon_0)s \leq r \leq (1 + \omega^2 \varepsilon_1)s. \tag{2.6}$$

With this notation,

$$2|a(u,u)| \geq |1 + t - r| + \omega^2 \varepsilon_i s.$$

Note that in the case $s \geq \frac{1}{2(1+\omega^2\varepsilon_1)}$, we have $|a(u,u)| \geq \frac{1}{2}\omega^2\varepsilon_i s \geq \frac{\omega^2\varepsilon_i}{4(1+\omega^2\varepsilon_1)}$. Otherwise, $s < \frac{1}{2(1+\omega^2\varepsilon_1)}$ so that $r < \frac{1}{2}$, and $|a(u,u)| \geq \frac{1}{2}|1+t-r| > \frac{1}{4}$. Hence, for all $s,t \geq 0$, and all r satisfying (2.6),

$$|a(u,u)| \geq c = \min\left\{\frac{\omega^2\varepsilon_i}{4(1+\omega^2\varepsilon_1)}, \frac{1}{4}\right\}.$$

The bound thus holds for every u with $\|u\|_{H^1(\Omega)} = 1$ and for every $\varepsilon \in \mathcal{A}$ with $\varepsilon_i > 0$. Given this coercivity bound, direct application of the Lax-Milgram Theorem yields existence of the bounded solution operator A^{-1} for problem (2.5) such that $\|A^{-1}\| \leq 1/c$. Thus $\|u\|_{H^1(\Omega)} \leq \|b\|_{H^1(\Omega)}/c$.

Given the bound on $\|u\|_{H^1(\Omega)}$, a uniform $H^2(\Omega)$ bound follows easily, since $\triangle u = -\omega^2\varepsilon u$ is uniformly bounded in $L^2(\Omega)$. □

LEMMA 2.2. *There exists a constant K such that for every $\varepsilon_s, \varepsilon_t \in \mathcal{A}$, if $u_s(\varepsilon_s)$, $u_t(\varepsilon_t)$ are the corresponding solutions of the Helmholtz equation (2.2)-(2.3), then u_s and u_t satisfy the Lipschitz condition:*

$$\|u_t - u_s\|_{H^2} \leq K\|\varepsilon_t - \varepsilon_s\|_{L^2}. \tag{2.7}$$

Moreover, there exists a constant C such that,

$$\|u_t - u_s\|_{L^\infty} \leq CK\|\varepsilon_t - \varepsilon_s\|_{L^2} \tag{2.8}$$

and

$$\|\nabla u_t - \nabla u_s\|_{L^\infty} \leq CK\|\varepsilon_t - \varepsilon_s\|_{L^2}. \tag{2.9}$$

Proof. We subtract one of the Helmholtz equations from the other to obtain:

$$\triangle u_t - \triangle u_s + \omega^2\varepsilon_t u_t - \omega^2\varepsilon_s u_s = 0.$$

Subtract $\omega^2\varepsilon_t u_s$ on both sides:

$$\triangle(u_t - u_s) + \omega^2\varepsilon_t(u_t - u_s) = -\omega^2(\varepsilon_t - \varepsilon_s)u_s.$$

Let $w = u_t - u_s$. Thus the above equation is written as:

$$\triangle w + \omega^2\varepsilon_t w = -\omega^2(\varepsilon_t - \varepsilon_s)u_s \tag{2.10}$$

The function $-\omega^2(\varepsilon_t - \varepsilon_s)u_s \in L^2(\Omega)$ and thus Lemma 2.1 applies and w is a solution to our equation (2.10). Let us rewrite (2.10) using the operator L_{ε_t}:

$$L_{\varepsilon_t} w := \triangle w + \omega^2\varepsilon_t w = -\omega^2(\varepsilon_t - \varepsilon_s)u_s.$$

From Lemma 2.1, the inverse operator $L_{\varepsilon_t}^{-1}\colon L^2(\Omega) \to H^2(\Omega)$ exists and is uniformly bounded with respect to $\varepsilon_t \in \mathcal{A}$. Thus,

$$w = -\omega^2 L_{\varepsilon_t}^{-1}(\varepsilon_t - \varepsilon_s)u_s.$$

Both when we have two- and three- dimensional materials, Sobolev Imbedding Theorem implies that $H^2(\Omega) \subset C_B^0(\Omega)$ [1] and

$$\|w\|_{H^2} \leq \|L_{\varepsilon_t}^{-1}\|_{L^2(\Omega),H^2(\Omega)}\|\varepsilon_t - \varepsilon_s\|_{L^2}\|u_s\|_{L^\infty}.$$

From here we see that:

$$\|u_t - u_s\|_{H^2} \le K\|\varepsilon_t - \varepsilon_s\|_{L^2}.$$

To prove the second part of the Lemma, we use Sobolev Imbedding Theorem and interpolation inequalities. We prove that $w \in W^{2,q}$ for any q such that $3 < q < \infty$. From the interpolation inequalities [1] we see that for any solution u of (2.2)-(2.3)

$$\|\triangle u\|_{L^q} \le \|\triangle u\|_{L^2}^{2/q}\|\triangle u\|_{L^\infty}^{1-2/q} \le \omega^2\|u\|_{H^2}^{2/q}\|\varepsilon u\|_{L^\infty}^{1-2/q} \le \omega^2\varepsilon_1^{1-2/q}\|u\|_{H^2}.$$

Thus $u \in W^{2,q}$. But Sobolev Imbedding Theorem [1] implies that $W^{2,q}(\Omega) \subset C_B^1(\Omega)$, i.e. there exists a constant C such that

$$\|u_t - u_s\|_{1,\infty} \le C\|u_t - u_s\|_{W^{2,q}} \le CK\|\varepsilon_t - \varepsilon_s\|_{L^2}, \tag{2.11}$$

where

$$\|u\|_{1,\infty} := \max_{0\le|\alpha|\le 1}\sup_{x\in\Omega}|D^\alpha u(x)|.$$

We deduce the Lipschitz conditions (2.8) and (2.9) from (2.11). □

We also obtain a Lipshitz-type bound that estimates the proximity of solutions u of the Helmholtz equation (2.2)-(2.3) and the solution $\tilde{u}$ of the constant coefficient Helmholtz equation, where the constant coefficient is the expected value of ε, i.e. $\tilde{\varepsilon} \equiv \langle\varepsilon\rangle = \varepsilon_0 p + \varepsilon_1(1-p)$. The bound is in terms of the local proximity of the random medium ε and the homogeneous medium $\tilde{\varepsilon}$.

LEMMA 2.3. *Let $\tilde{u}$ be the solution to the Helmholtz equation with constant coefficient $\tilde{\varepsilon} = \varepsilon_0 p + \varepsilon_1(1-p)$, still satisfying boundary condition (2.3):*

$$\triangle\tilde{u} + \omega^2\tilde{\varepsilon}\tilde{u} = 0. \tag{2.12}$$

Let $\nu > 0$ and $3 < q < \infty$ be fixed. For any subdomain $\tilde{\Omega} \subset \Omega$ we define the diameter

$$d(\tilde{\Omega}) = \sup_{x,y\in\tilde{\Omega}}|x-y|.$$

There exist constants K^* and K^*_∞, and $\gamma > 0$ such that if Ω is divided into N' non-overlapping subdomains O_i such that $d(O_i) \le \gamma$ for all $i = 1,\ldots,N'$, then

$$\|u - \tilde{u}\|_{L^2} \le K^*\left(\sum_{i=1}^{N'}\left|\int_{O_i}(\tilde{\varepsilon} - \varepsilon)\,dx\right|\right) + \nu \tag{2.13}$$

and

$$\|u - \tilde{u}\|_{L^\infty} \le K^*_\infty(q)\left(K^*\left(\sum_{i=1}^{N'}\left|\int_{O_i}(\tilde{\varepsilon} - \varepsilon)\,dx\right|\right) + C\nu\right)^{\frac{1}{q}}, \tag{2.14}$$

for all realizations (u,ε), and $3 < q < \infty$.

Proof. Subtract the two equations (2.1) and (2.12) and manipulate them to get the equation:

$$\triangle(u - \tilde{u}) + \omega^2\tilde{\varepsilon}(u - \tilde{u}) = \omega^2(\tilde{\varepsilon} - \varepsilon)u$$

for any realization (ε, u). Thus, we can apply the solution operator $L_{\tilde{\varepsilon}}^{-1}$ to obtain:

$$u - \tilde{u} = \omega^2 L_{\tilde{\varepsilon}}^{-1}((\tilde{\varepsilon} - \varepsilon)u).$$

The $L_{\tilde{\varepsilon}}^{-1}$ is a bounded operator $L_{\tilde{\varepsilon}}^{-1} : L^2 \to H^2$ and a compact operator $L_{\tilde{\varepsilon}}^{-1} : L^2 \to L^2$. Since $L_{\tilde{\varepsilon}}^{-1} : L^2 \to L^2$ is compact, it can be approximated by a sequence of finite-rank operators L_n^{-1}, and for every given error $\nu_1 > 0$, there exists M_1 such that $\|L_{\tilde{\varepsilon}}^{-1} - L_n^{-1}\|_{L^2(\Omega),L^2(\Omega)} \leq \nu_1$ for $n \geq M_1$ [4]. We apply the triangular inequality to obtain:

$$\begin{aligned}
&\|u - \tilde{u}\|_{L^2} = \omega^2 \|L_{\tilde{\varepsilon}}^{-1}(\tilde{\varepsilon} - \varepsilon)u\|_{L^2} \\
&\leq \omega^2 \|L_{\tilde{\varepsilon}}^{-1} - L_n^{-1}\|_{L^2(\Omega),L^2(\Omega)} \|\tilde{\varepsilon} - \varepsilon\|_{L^\infty} \|u\|_{L^2} + \omega^2 \|L_n^{-1}(\tilde{\varepsilon} - \varepsilon)u\|_{L^2} \\
&\leq C\nu_1 + \omega^2 \|L_n^{-1}(\tilde{\varepsilon} - \varepsilon)u\|_{L^2},
\end{aligned}$$

where C is independent of material ε. Finite-rank operators can be decomposed

$$L_n^{-1}(\tilde{\varepsilon} - \varepsilon)u = \sum_{i=1}^{N} w_i^n \langle(\tilde{\varepsilon} - \varepsilon)u, g_i^n\rangle_{L^2}$$

where $g_i^n \in L^2(\Omega)$ and $w_i^n \in \text{Range}(L_n^{-1})$. Thus,

$$\|L_n^{-1}(\tilde{\varepsilon} - \varepsilon)u\|_{L^2} = \|\sum_{i=1}^{N} w_i^n \int_\Omega (\tilde{\varepsilon} - \varepsilon)u g_i^n \, dx\|_{L^2} \leq \sum_{i=1}^{N} \|w_i^n\|_{L^2} \left| \int_\Omega (\tilde{\varepsilon} - \varepsilon)u g_i^n \, dx \right|.$$

Fix $n \geq M_1$; g_i^n is a measurable function on Ω. Given $\nu_2 \geq 0$, there exist continuous functions v_i^n on Ω such that $|S_{\nu_2}| = m\{x : g_i^n(x) \neq v_i^n(x)\} \leq \nu_2$, for each $i = 1, \ldots, N$ [17]. Decompose the integral

$$\int_\Omega (\tilde{\varepsilon} - \varepsilon)u g_i^n \, dx = \int_{\Omega \setminus S_{\nu_2}} (\tilde{\varepsilon} - \varepsilon)u g_i^n \, dx + \int_{S_{\nu_2}} (\tilde{\varepsilon} - \varepsilon)u g_i^n \, dx.$$

Using this we obtain the following bound for each $i = 1, \ldots, N$

$$\begin{aligned}
&\left| \int_\Omega (\tilde{\varepsilon} - \varepsilon)u g_i^n \, dx \right| \leq \left| \int_{\Omega \setminus S_{\nu_2}} (\tilde{\varepsilon} - \varepsilon)u g_i^n \, dx \right| + \left| \int_{S_{\nu_2}} (\tilde{\varepsilon} - \varepsilon)u g_i^n \, dx \right| \\
&\leq \left| \int_{\Omega \setminus S_{\nu_2}} (\tilde{\varepsilon} - \varepsilon)u g_i^n \, dx \right| + \|\tilde{\varepsilon} - \varepsilon\|_{L^\infty} |S_{\nu_2}|^{\frac{1}{2}} \|u\|_{L^\infty} \|g_i^n\|_{L^2} \leq \left| \int_{\Omega \setminus S_{\nu_2}} (\tilde{\varepsilon} - \varepsilon)u g_i^n \, dx \right| + C\nu_2.
\end{aligned}$$

The function v_i^n is continuous on the compact domain Ω and thus it is uniformly continuous and can be approximated by a sequence of step functions $\psi_{N'}$. Divide Ω into N' non-overlapping subdomains O_i such that $d(O_i) \leq \gamma$. Define $\psi_{N'} = \sum_{i=1}^{N'} a_i^{N'} \chi_{O_i}$, where χ_{O_i} is a characteristic function of the subdomain O_i. For every given error

$\nu_3 > 0$, there exists $\gamma > 0$ such that $\|v_i^n - \psi_{N'}\|_{L^\infty} \leq \nu_3$. Thus,

$$\left|\int_{\Omega\setminus S_{\nu_2}} (\tilde{\varepsilon}-\varepsilon) u g_i^n \, dx\right| = \left|\int_{\Omega\setminus S_{\nu_2}} (\tilde{\varepsilon}-\varepsilon) u v_i^n \, dx\right| = \left|\int_{\Omega} (\tilde{\varepsilon}-\varepsilon) u v_i^n \, dx - \int_{S_{\nu_2}} (\tilde{\varepsilon}-\varepsilon) u v_i^n \, dx\right|$$
$$\leq \left|\int_{\Omega} (\tilde{\varepsilon}-\varepsilon) u v_i^n \, dx\right| + \|\tilde{\varepsilon}-\varepsilon\|_{L^\infty} \, |S_{\nu_2}| \, \|u\|_{L^\infty} \|v_i^n\|_{L^\infty}$$
$$\leq \left|\int_{\Omega} (\tilde{\varepsilon}-\varepsilon) u (v_i^n - \psi_{n'}) \, dx\right| + \left|\int_{\Omega} (\tilde{\varepsilon}-\varepsilon) u \psi_{N'} \, dx\right| + C\nu_2$$
$$\leq \left|\int_{\Omega} (\tilde{\varepsilon}-\varepsilon) u \psi_{N'} \, dx\right| + \|v_i^n - \psi_{N'}\|_{L^\infty} \|\tilde{\varepsilon}-\varepsilon\|_{L^1} \|u\|_{L^\infty} + C\nu_2$$
$$\leq \left|\int_{\Omega} (\tilde{\varepsilon}-\varepsilon) u \sum_{i=1}^{N'} a_i^{N'} \chi_{O_i} \, dx\right| + C\nu_3 + C\nu_2 \leq \sum_{i=1}^{N'} |a_i^{N'}| \left|\int_{O_i} (\tilde{\varepsilon}-\varepsilon) u \, dx\right| + C\nu_3 + C\nu_2.$$

From Lemma 2.2, there exists a constant K such that $\|u\|_{H^2} \leq K$ for every realization u. Since H^2 imbeds in $C^{0,1/2}$, there exists a constant K_L such that

$$|u(x) - u(y)| \leq K_L |x-y|^{1/2},$$

for all u and for all $x, y \in \Omega$. Let

$$u_\gamma^i = \max_{x \in O_i} u(x)$$

and we have

$$|u(x) - u_\gamma^i| \leq K_L \gamma^{1/2}$$

for all $x \in O_i$. Thus,

$$\left|\int_{\Omega\setminus S_{\nu_2}} (\tilde{\varepsilon}-\varepsilon) u g_i^n \, dx\right|$$
$$\leq \sum_{i=1}^{N'} |a_i^{N'}| \left|\int_{O_i} (\tilde{\varepsilon}-\varepsilon)(u - u_\gamma^i) \, dx\right| + \sum_{i=1}^{N'} |a_i^{N'}| \left|\int_{O_i} (\tilde{\varepsilon}-\varepsilon)(u_\gamma^i) \, dx\right| + C\nu_3 + C\nu_2$$
$$\leq K_L \gamma^{1/2} \sum_{i=1}^{N'} |a_i^{N'}| \left|\int_{O_i} (\tilde{\varepsilon}-\varepsilon) \, dx\right| + \sum_{i=1}^{N'} |a_i^{N'}| \left|\int_{O_i} (\tilde{\varepsilon}-\varepsilon)(u_\gamma^i) \, dx\right| + C\nu_3 + C\nu_2$$
$$\leq C\gamma^{1/2} + \sum_{i=1}^{N'} |a_i^{N'}| |u_\gamma^i| \left|\int_{O_i} (\tilde{\varepsilon}-\varepsilon) \, dx\right| + C\nu_3 + C\nu_2.$$

We obtain the desired bound by taking γ sufficiently small. Hence,

$$\|u - \tilde{u}\|_{L^2} \leq K^* \left(\sum_{i=1}^{N'} \left|\int_{O_i} (\tilde{\varepsilon}-\varepsilon) \, dx\right| \right) + \nu. \tag{2.15}$$

The interpolation inequality [1] states that there exists a constant K_I such that

$$\|u\|_{W^{1,q}} \leq K_I \|u\|_{W^{2,q}}^{\frac{1}{2}} \|u\|_{L^q}^{\frac{1}{2}}.$$

Since $W^{1,q}$ imbeds in C_B for $3 < q < \infty$, there exists a constant C such that

$$\|u-\tilde{u}\|_{L^\infty} \le C\|u-\tilde{u}\|_{W^{1,q}}.$$

Also, the interpolation inequality for L^p-spaces [7] states that when $3 < q < \infty$

$$\|u\|_{L^q} \le \|u\|_{L^2}^{\frac{2}{q}} \|u\|_{L^\infty}^{\frac{q-2}{q}}.$$

Combining the above inequalities and the bound (2.15), we prove the second bound in the statement of the Lemma:

$$\|u-\tilde{u}\|_{L^\infty} \le CK_I\|u-\tilde{u}\|_{W^{2,q}}^{\frac{1}{2}}\|u-\tilde{u}\|_{L^q}^{\frac{1}{2}} \le CK_I\|u-\tilde{u}\|_{W^{2,q}}^{\frac{1}{2}}\|u-\tilde{u}\|_{L^2}^{\frac{1}{q}}\|u-\tilde{u}\|_{L^\infty}^{\frac{q-2}{2q}}$$
$$\le K_\infty^*(q)\left(K^*\left(\sum_{i=1}^{N'}\left|\int_{O_i}(\tilde{\varepsilon}-\varepsilon)\,dx\right|\right)+C\nu\right)^{\frac{1}{q}}.$$

□

3. Effective Dielectric Coefficient. The expected value $\langle u\rangle$ of the solution u of the Helmholtz equation (2.2)-(2.3), that depends on the random variables through its dependence on the composite material, is defined as follows

$$\langle u\rangle = \int_{\Psi_\delta} u\,dP = \int_{\Gamma_\delta}\sum_{m_g\in R_g}\prod_{j=1}^{N_g} p^{1-m_j}\,(1-p)^{m_j}\,u(\varepsilon_{m,g},x)\,dG_\delta. \tag{3.1}$$

Note that $\langle\rangle$ is an expectation over material realizations, not the spatial variables, so that $\langle u\rangle$ is in general still a function of x. Thus, the effective dielectric coefficient, defined in (1.2) as

$$\varepsilon^* = \frac{\langle\varepsilon u\rangle}{\langle u\rangle},$$

is a function of the spatial variable x.

Our main theorem gives a bound on the dielectric coefficient and its spatial variations provided we have a lower bound on the expected value of u. Such a bound is proven to exist for sufficiently small δ. The theorem shows that as the maximum volume δ of the subdomains decreases, so does the magnitude of the spatial variations, and as $\delta \to 0$, the effective coefficient equals the constant predicted by the quasistatic case.

THEOREM 3.1. *Let $\varepsilon^*(x)$ be the effective dielectric coefficient of the medium defined by (1.2). For sufficiently small δ, $\varepsilon^*(x)$ is bounded from above uniformly for all $x \in \Omega$. For such δ there exists a constant C^* such that $\|\varepsilon^*\|_{BV} \le C^*|\varepsilon_1-\varepsilon_0|\delta$, and the spatial variations of $\varepsilon^*(x)$ are bounded in terms of the size of the inhomogeneities δ and the contrast of the medium $|\varepsilon_1-\varepsilon_0|$. As the size of the inhomogeneities goes to 0, the spatial variations decrease in magnitude, and $\varepsilon^*(x) \to p\varepsilon_0 + (1-p)\varepsilon_1$.*

Proof. The proof applies to one-, two-, and three-dimensional random media. In order to obtain a bound on $|\varepsilon^*| = \frac{|\langle\varepsilon u\rangle|}{|\langle u\rangle|}$, we must obtain a lower bound on the denominator $|\langle u\rangle|$. A uniform bound exists provided δ is chosen sufficiently small, i.e. $|\langle u\rangle| \ge c > 0$ for all $x \in \Omega$. The proof is based on a probability argument that shows that the probability that the solutions u will be within a certain radius

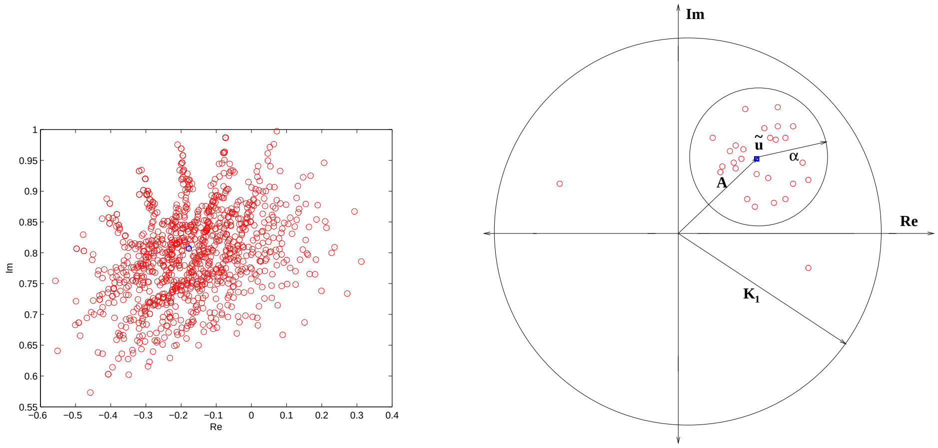

FIG. 3.1. *Proximity to the constant coefficient solution. Left: From numerical experiments, solutions u for a medium with 10 layers at $x = 0.5$ (red dots) and the solution to the constant coefficient problem $\tilde{u}(0.5)$ (blue square); Right: For appropriate parameter δ, the probability that solutions u cluster within a circle with center $\tilde{u}$ and radius α is $1-\beta$. The probability β that solutions lay outside this circle depends on δ, and $\beta \to 0$ as $\delta \to 0$. All solutions are contained in the circle with radius K_1, since $\|u\|_{L^\infty} \leq K_1$.*

α from the solution of the constant boundary value problem with dielectric constant $\tilde{\varepsilon} = p\varepsilon_0 + (1-p)\varepsilon_1$ goes to one as the maximum volume δ or the contrast $|\varepsilon_1 - \varepsilon_0|$ goes to zero. The probability β that a solution u lies outside the circle with radius α depends on the parameter δ, and $\beta \to 0$ as $\delta \to 0$. This prevents $\langle u \rangle$ to equal 0 and gives a lower bound on $|\langle u \rangle| \geq c > 0$. The numerical experiment in Figure 3.1 illustrates this argument, and the proof follows.

We let α and β be arbitrary constants such that $\beta \leq 1$ and $\alpha \leq K_1$. We want to prove that for every such α and β, one can find $\delta > 0$ such that $|\Omega_j| \leq \delta$ for all $j = 1, \ldots, N$ and

$$|\langle u \rangle| \geq (1-\beta)(A-\alpha) - \beta K_1,$$

where $\|\tilde{u}\|_{L^\infty} = A$ and $\|u\|_{L^\infty} \leq K_1$.

We use Lemma 2.3. There our domain Ω was divided into N' non-overlapping subdomains O_i such that $d(O_i) \leq \gamma$ for all $i = 1, \ldots, N'$. Now divide each O_i into subdomains $\{O_i^j\}_{j=1}^{N_\delta}$ such that $|O_i^j| \leq \delta$ and $\cup O_i^j = O_i$. Given radius α and using Chebyshev's inequality [6] and estimate (2.14), we obtain

$$P(\|u-\tilde{u}\|_{L^\infty} \leq \alpha) = 1 - P(\|u-\tilde{u}\|_{L^\infty} \geq \alpha) \geq 1 - \frac{1}{\alpha^q}\left\langle \|u-\tilde{u}\|_{L^\infty}^q \right\rangle \tag{3.2}$$
$$\geq 1 - \left(\frac{K_\infty^*}{\alpha}\right)^q K^* \left(\sum_{i=1}^{N'} \left\langle \left| \int_{O_i} (\tilde{\varepsilon} - \varepsilon)\, dx \right| \right\rangle \right) + C\nu \equiv 1-\beta$$

Now, we want to estimate $\left\langle \left| \int_{O_i} (\tilde{\varepsilon} - \varepsilon)\, dx \right| \right\rangle$. We calculate that $(\tilde{\varepsilon} - \varepsilon_1) = -p(\varepsilon_1 - \varepsilon_0)$ and $(\tilde{\varepsilon} - \varepsilon_0) = (1-p)(\varepsilon_1 - \varepsilon_0)$. The smallest number of subdomains will occur when δ divides $|O_i|$ and $|O_i|\delta = N_\delta$. Because we have absolute values in $\left\langle \left| \int_{O_i} (\tilde{\varepsilon} - \varepsilon)\, dx \right| \right\rangle$ the expected value for this fixed number of subdomains will be affected by the value of p. The probability p falls in one of N_δ different intervals $I_1 \equiv \left[0, \frac{\delta}{|O_i|}\right), I_2 \equiv \left[\frac{\delta}{|O_i|}, \frac{2\delta}{|O_i|}\right), \ldots, I_{N_\delta} \equiv \left[\frac{(N_\delta - 1)\delta}{|O_i|}, 1\right]$. Let t denote the number of the interval where p falls. Let

$$f(t; N_\delta, p) = \binom{N_\delta}{t} p^t (1-p)^{N_\delta - t}$$

for $t = 0, 1, 2, \ldots, N_\delta$ and where $\binom{N_\delta}{t} = \frac{N_\delta!}{t!(N_\delta - t)!}$. Now we can find the conditional expected values $\left\langle \left| \int_{O_i} (\tilde{\varepsilon} - \varepsilon)\, dx \right| : |O_i|\delta = N_\delta \right\rangle$ for p in the interval I_t:

$$\left\langle \left| \int_{O_i} (\tilde{\varepsilon} - \varepsilon)\, dx \right| : |O_i|\delta = N_\delta \right\rangle = 2|\varepsilon_1 - \varepsilon_0| \delta f(t; N_\delta, p)\kappa,$$

where

$$\kappa = \begin{cases} p \times \#\text{intervals } I_i \text{ such that } p_i \le p_t & \text{if } p < 1 - p; \\ (1-p) \times \#\text{intervals } I_i \text{ such that } p_i \le p_t & \text{otherwise.} \end{cases}$$

We use the Stirling's formula $n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\lambda_n}, \quad \frac{1}{12n+1} < \lambda_n < \frac{1}{12n}$, in order to prove that the sequence $\binom{N_\delta}{t} p^t (1-p)^{N_\delta - t}$ converges to 0 for every $p \in I_t$ and that $\left\langle \left| \int_{O_i} (\tilde{\varepsilon} - \varepsilon)\, dx \right| : |O_i|\delta = N_\delta \right\rangle \to 0$ as $\delta \to 0$, or equivalently as $N_\delta \to \infty$. In order to bound $\left\langle \left| \int_{O_i} (\tilde{\varepsilon} - \varepsilon)\, dx \right| \right\rangle$ we must consider all possible realization. The bound is obtained by adding terms as above where N_δ is replaced by the appropriate number of intervals and the intervals $I_1, I_2, \ldots$ will depend on the volumes of the subdomains in which O_i is divided. Nevertheless, the same argument as above will apply to show that $\left\langle \left| \int_{O_i} (\tilde{\varepsilon} - \varepsilon)\, dx \right| \right\rangle \to 0$ as $\delta \to 0$ or as the contrast $|\varepsilon_1 - \varepsilon_0| \to 0$.

We have shown that the probability that solutions u are within radius α of the constant coefficient solution $\tilde{u}$ goes to one as either δ or the contrast in the media $|\varepsilon_1 - \varepsilon_0|$ goes to 0.

Let us call $\|u - \tilde{u}\|_{L^\infty} \le \alpha$ condition L and the complement - condition L^c. Define the conditional expectations

$$\langle u | L \rangle \equiv \frac{\int_{\Psi_\delta(L)} u\, dP}{P(L)} \quad \text{and} \quad \langle u | L^c \rangle \equiv \frac{\int_{\Psi_\delta(L^c)} u\, dP}{P(L^c)},$$

and note that $P(L) = 1 - \beta$ and $P(L^c) = \beta$. The expected value $\langle u \rangle$ is given by

$$\langle u \rangle = P(L)\langle u | L \rangle + P(L^c)\langle u | L^c \rangle,$$

and using estimate (3.2) we obtain

$$|\langle u \rangle| \ge (1 - \beta)|\langle u | L \rangle| - \beta |\langle u | L^c \rangle|.$$

If u satisfies (3.2), then u satisfies the inequality

$$\|u\|_{L^\infty} \ge \|\tilde{u}\|_{L^\infty} - \alpha \ge A - \alpha.$$

And now using the uniform upper bound $\|u\|_{L^\infty} \le K_1$, we obtain the desired result:

$$|\langle u \rangle| \ge (1 - \beta)(A - \alpha) - \beta K_1,$$

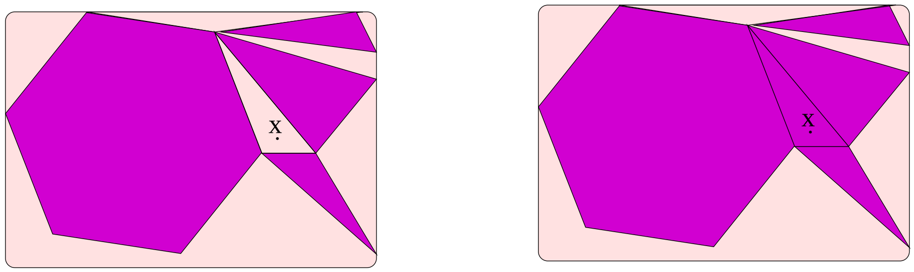

FIG. 3.2. *Sample materials in Ψ_δ^0 and Ψ_δ^1 for fixed x. Left: Material realization ψ_0; Right: Corresponding material realization ψ_1 obtained by switching material ε_0 with material ε_1 in the domain containing x.*

where the constant β depends on δ,the maximum volume of the subdomains, and on the contrast $|\varepsilon_1 - \varepsilon_0|$, and $\beta \to 0$ as δ or $|\varepsilon_1 - \varepsilon_0| \to 0$. Thus by picking the appropriate α and β, where β is controlled by the parameter δ, we obtain the lower bound $|\langle u \rangle| \geq c > 0$ for all $x \in \Omega$. This provides a bound on the effective dielectric coefficient:

$$|\varepsilon^*| \leq \frac{\tilde{\varepsilon} K_1}{c}.$$

The uniform lower bound on $|\langle u \rangle|$ is utilized in proving that $\|\varepsilon^*\|_{BV} \leq C^* |\varepsilon_1 - \varepsilon_0| \delta$. Formally, the gradient $\nabla \varepsilon^*$ is given by:

$$\nabla \varepsilon^* = \frac{\langle u \rangle \langle (\nabla \varepsilon) u \rangle + \langle u \rangle \langle \varepsilon \nabla u \rangle - \langle \nabla u \rangle \langle \varepsilon u \rangle}{\langle u \rangle^2}, \tag{3.3}$$

where $\nabla \varepsilon$ is understood in the sense of a distribution. Now choose δ such that $|\langle u \rangle| \geq c > 0$. We want to bound the numerator in terms of this δ and the contrast $|\varepsilon_1 - \varepsilon_0|$. First we bound

$$|\langle u \rangle \langle \varepsilon \nabla u \rangle - \langle \nabla u \rangle \langle \varepsilon u \rangle| \leq C_1 \delta |\varepsilon_1 - \varepsilon_0| \tag{3.4}$$

pointwise, where C_1 is a constant. In the proof we use the Lipschitz bounds (2.8) and (2.9) from Lemma 2.2.

And after we prove that

$$|\langle u \rangle \langle (\nabla \varepsilon) u \rangle| \leq C_2 \delta |\varepsilon_1 - \varepsilon_0|,$$

where C_2 is a constant and $\nabla \varepsilon$ is understood in the distributional sense, since the regular gradient is undefined for x on the interface between two or more subdomains with alternating materials ε_1 and ε_0 in them.

The bound (3.4) is obtained by looking at material realizations that differ only in the subdomain $\Omega_j \ni x$ and realizing that the pointwise difference in solutions propagating through two such material realizations can be bounded in terms of the L^2-norm of the difference in the two materials, where the two materials differ only on subdomain Ω_j with $|\Omega_j| \leq \delta$.

Fix x. Divide Ψ_δ into two subsets $\Psi_\delta = \Psi_\delta^0 \cup \Psi_\delta^1$: Ψ_δ^0 is the subset of realizations such that $\varepsilon(x) = \varepsilon_0$ and Ψ_δ^1 is the subset of realizations such that $\varepsilon(x) = \varepsilon_1$. Sample materials in subsets Ψ_δ^0 and Ψ_δ^1 are shown in Figure 3.2. Let R_g^0 and R_g^1 be subsets of R_g such that

$$R_g^0 = \{m_g = \left(m_1, \ldots, m_{N_g}\right) : m_j = 0 \text{ for } x \in \Omega_j\},$$

and

$$R_g^1 = \{m_g = \left(m_1, \ldots, m_{N_g}\right) : m_j = 1 \text{ for } x \in \Omega_j\}.$$

Thus, $R_g = R_g^0 \cup R_g^1$. The expected value of u is given by:

$$\langle u \rangle = \int_{\Psi_\delta} u\, dP = \int_{\Gamma_\delta} \sum_{m_g \in R_g} \prod_{l=1}^{N_g} p^{1-m_l} \left(1-p\right)^{m_l} u(\varepsilon_{m,g}, x)\, dG_\delta$$

$$= p \int_{\Gamma_\delta} \sum_{m_g \in R_g^0} \prod_{\substack{l=1 \\ l \neq j}}^{N_g} p^{1-m_l} \left(1-p\right)^{m_l} u\, dG_\delta + (1-p) \int_{\Gamma_\delta} \sum_{m_g \in R_g^1} \prod_{\substack{l=1 \\ l \neq j}}^{N_g} p^{1-m_l} \left(1-p\right)^{m_l} u\, dG_\delta$$

$$= p\langle u|\varepsilon(x) = \varepsilon_0\rangle + (1-p)\langle u|\varepsilon(x) = \varepsilon_1\rangle = p\langle u\rangle_{\Psi_\delta^0} + (1-p)\langle u\rangle_{\Psi_\delta^1},$$

where $\langle u\rangle_{\Psi_\delta^0}$ is the conditional expectation of u given $\varepsilon(x) = \varepsilon_0$, and $\langle u\rangle_{\Psi_\delta^1}$ is the conditional expectation of u given $\varepsilon(x) = \varepsilon_1$. Using this notation we can rewrite

$$\langle u\rangle\langle\varepsilon\nabla u\rangle - \langle\nabla u\rangle\langle\varepsilon u\rangle$$
$$= \varepsilon_1 p(1-p) \left(\langle u\rangle_{\Psi_\delta^0}\langle\nabla u\rangle_{\Psi_\delta^1} - \langle u\rangle_{\Psi_\delta^1}\langle\nabla u\rangle_{\Psi_\delta^0}\right) + \varepsilon_0 p(1-p) \left(\langle u\rangle_{\Psi_\delta^1}\langle\nabla u\rangle_{\Psi_\delta^0} - \langle u\rangle_{\Psi_\delta^0}\langle\nabla u\rangle_{\Psi_\delta^1}\right).$$

For every material described by Ψ_δ^0, there exists a material described by Ψ_δ^1 such that the two materials differ only in a subdomain $\Omega_j \ni x$. Let us call u_{ψ_0} the solution of the Helmholtz equation when the material realization belongs to Ψ_δ^0 and u_{ψ_1} the corresponding solution of the Helmholtz equation when the material realization, differing only in m_j, belongs to Ψ_δ^1. We have

$$\left|\int_{\Psi_\delta^1} u_{\psi_1}(x)\, dP - \int_{\Psi_\delta^0} u_{\psi_0}(x)\, dP\right| \leq \int_{\Gamma_\delta} \sum_{i=1}^{2^{N_g-1}} \prod_{\substack{l=1 \\ l \neq j}}^{N_g} p^{1-m_l^i} \left(1-p\right)^{m_l^i} |u_{\psi_1} - u_{\psi_0}|(x)\, dG_\delta$$

$$\leq \sup_{\substack{g \in \Gamma_\delta \\ m_1 \in R_g^1 \\ m_0 \in R_g^0}} \|u_{\psi_1}(m_1, g) - u_{\psi_0}(m_0, g)\|_{L^\infty}$$

$$\leq CK \sup_{\substack{g \in \Gamma_\delta \\ m_1 \in R_g^1 \\ m_0 \in R_g^0}} \|\varepsilon_{\psi_1}(m_1, g) - \varepsilon_{\psi_0}(m_0, g)\|_{L^2} \leq CK\delta|\varepsilon_1 - \varepsilon_0|.$$

The preceding comes from the fact that for any material described by Ψ_δ^1, one can find a material described by Ψ_δ^0, which differs only on the subdomain $\Omega_j \ni x$ with volume less than or equal to δ and from application of Lemma 2.2. Thus, we have that $\langle u\rangle_{\Psi_\delta^1} = \langle u\rangle_{\Psi_\delta^0}$ pointwise as $\delta \to 0$. By a similar argument, $\left|\langle\nabla u\rangle_{\Psi_\delta^1} - \langle\nabla u\rangle_{\Psi_\delta^0}\right| \leq CK\delta|\varepsilon_1 - \varepsilon_0|$, and $\langle\nabla u\rangle_{\Psi_\delta^1} = \langle\nabla u\rangle_{\Psi_\delta^0}$ pointwise as $\delta \to 0$. Now,

$$\left|\langle u\rangle_{\Psi_\delta^0}\langle\nabla u\rangle_{\Psi_\delta^1} - \langle u\rangle_{\Psi_\delta^1}\langle\nabla u\rangle_{\Psi_\delta^0}\right| \tag{3.5}$$
$$\leq \left|\langle u\rangle_{\Psi_\delta^0}\right| \left|\langle\nabla u\rangle_{\Psi_\delta^1} - \langle\nabla u\rangle_{\Psi_\delta^0}\right| + \left|\langle\nabla u\rangle_{\Psi_\delta^0}\right| \left|\langle u\rangle_{\Psi_\delta^1} - \langle u\rangle_{\Psi_\delta^0}\right|.$$

From Lemmas 2.2 and 2.1, we know that $u \in C_B^1(\Omega)$, and that there exist constants K_1 and K_2 such that $\|u\|_{L^\infty} \leq K_1$ and $\|\nabla u\|_{L^\infty} \leq K_2$ for every u. Then

$$\left|\langle u\rangle_{\Psi_\delta^0}\langle\nabla u\rangle_{\Psi_\delta^1} - \langle u\rangle_{\Psi_\delta^1}\langle\nabla u\rangle_{\Psi_\delta^0}\right| \leq KC|\varepsilon_1 - \varepsilon_0|\delta(K_1 + K_2) \to 0 \quad \text{as} \quad \delta \to 0$$

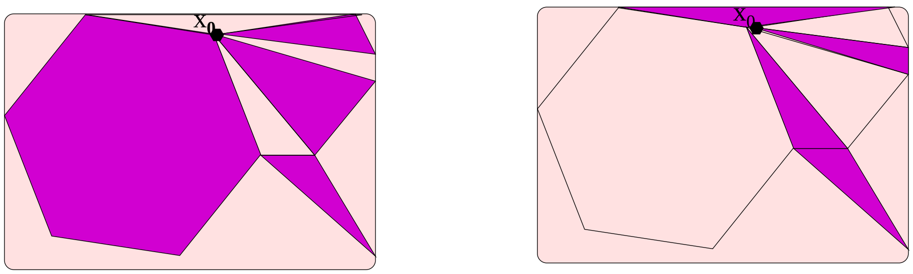

FIG. 3.3. *Sample materials in Ψ_δ^α and Ψ_δ^β for fixed x on the boundary between several materials. Left: Material realization ψ_α; Right: Corresponding material realization ψ_β obtained by interchanging the materials at domains interfacing at x.*

and similarly for the second term in (3.5). Thus, we obtain the following bound

$$
\begin{aligned}
&|\langle u\rangle\langle\varepsilon\nabla u\rangle - \langle\nabla u\rangle\langle\varepsilon u\rangle| \qquad (3.6)\\
&\le \varepsilon_1 p(1-p)\left|\langle u\rangle_{\Psi_\delta^0}\langle\nabla u\rangle_{\Psi_\delta^1} - \langle u\rangle_{\Psi_\delta^1}\langle\nabla u\rangle_{\Psi_\delta^0}\right| + \varepsilon_0 p(1-p)\left|\langle u\rangle_{\Psi_\delta^1}\langle\nabla u\rangle_{\Psi_\delta^0} - \langle u\rangle_{\Psi_\delta^0}\langle\nabla u\rangle_{\Psi_\delta^1}\right|\\
&\le KCp(1-p)(\varepsilon_1+\varepsilon_0)|\varepsilon_1-\varepsilon_0|(K_1+K_2)\delta \to 0 \quad \text{as} \quad \delta\to 0.
\end{aligned}
$$

Now, we want to prove that $|\langle(\nabla\varepsilon)u\rangle| \le C_2\delta|\varepsilon_1-\varepsilon_0|$ in the distributional sense. Assume that each Ω_j has finite perimeter, i.e. $\|\chi_{\Omega_j}\|_{BV} \le \delta$, where χ_{Ω_j} is the characteristic function of the set. This condition excludes from consideration materials containing a subdomain with infinite perimeter. Assume that there exists η such that $0 < \eta \le \delta$ and $B_\eta(x)$ intersects at most k subdomains Ω_j for all $x \in \Omega$. This condition excludes from consideration materials with infinitely many subdomains interfacing at any $x \in \Omega$. Also assume that $\sum_{j=1}^k \|\chi_{\Omega_j}\|_{BV} \le C_p$ for all geometries and a constant C_p that is independent of the geometries. Since $\varepsilon(x)$ equals a constant in every subdomain Ω_j, $\nabla\varepsilon = 0$ there, and the only problem occurs at the interface between two or more subdomains with different materials, where ε is discontinuous and $\nabla\varepsilon$ is undefined in the regular sense. We define $\nabla\varepsilon$ in the distributional sense.

Fix a realization ψ_α such that x_0 is at the interface between k subdomains Ω_j, $j = 1...k$ with alternating materials ε_0 and ε_1 in them. This assumption will pose no loss of generality since the other cases are attained at material realizations satisfying our assumptions. Call ψ_β the realization that has the same geometry as realization ψ_α, but with the materials in the k subdomains interfacing at x_0 switched, e.g. Figure 3.3. Without loss of generality let realization ψ_α have material ε_0 in Ω_1; thus realization ψ_β has material ε_1 in the same subdomain Ω_1. Let ϕ be a test function $\phi \in C_0^\infty(\Omega, \mathbb{R}^n)$ such that $\operatorname{supp}\phi \in B_\eta(x_0)$. We can find $\nabla(\varepsilon_\alpha)u_\alpha$ at x_0 in the generalized sense:

$$
\begin{aligned}
&\int_{B_\eta(x_0)} \nabla(\varepsilon_\alpha)u_\alpha\phi\,dx\\
&= (\varepsilon_1-\varepsilon_0)\int_{\partial(\Omega_1\cap\Omega_2)} u_\alpha\phi\nu_{\partial(\Omega_1\cap\Omega_2)}\,dx + (\varepsilon_1-\varepsilon_0)\int_{\partial(\Omega_2\cap\Omega_3)} u_\alpha\phi\nu_{\partial(\Omega_2\cap\Omega_3)}\,dx + \cdots\\
&+(\varepsilon_1-\varepsilon_0)\int_{\partial(\Omega_{k-1}\cap\Omega_k)} u_\alpha\phi\nu_{\partial(\Omega_{k-1}\cap\Omega_k)}\,dx + (\varepsilon_1-\varepsilon_0)\int_{\partial(\Omega_1\cap\Omega_k)} u_\alpha\phi\nu_{\partial(\Omega_1\cap\Omega_k)}\,dx,
\end{aligned}
$$

where $\partial(\Omega_1\cap\Omega_2)$ is the interface between subdomains Ω_1 and Ω_2 and $\nu_{\partial(\Omega_1\cap\Omega_2)}$ is the unit normal vector to Ω_1 on the interface with Ω_2. Note that $\nu_{\partial(\Omega_1\cap\Omega_2)} = -\nu_{\partial(\Omega_2\cap\Omega_1)}$.

Similarly, we find that $\nabla(\varepsilon_\beta)u_\beta$ at x_0 in the generalized sense is

$$\begin{aligned}&\int_{B_\eta(x_0)} \nabla(\varepsilon_\beta)u_\beta\phi\,dx\\ &= -(\varepsilon_1-\varepsilon_0)\int_{\partial(\Omega_1\cap\Omega_2)} u_\beta\phi\nu_{\partial(\Omega_1\cap\Omega_2)}\,dx - (\varepsilon_1-\varepsilon_0)\int_{\partial(\Omega_2\cap\Omega_3)} u_\beta\phi\nu_{\partial(\Omega_2\cap\Omega_3)}\,dx - \cdots\\ &-(\varepsilon_1-\varepsilon_0)\int_{\partial(\Omega_{k-1}\cap\Omega_k)} u_\beta\phi\nu_{\partial(\Omega_{k-1}\cap\Omega_k)}\,dx - (\varepsilon_1-\varepsilon_0)\int_{\partial(\Omega_1\cap\Omega_k)} u_\beta\phi\nu_{\partial(\Omega_1\cap\Omega_k)}\,dx\end{aligned}$$

Divide again Ψ_δ into three subsets $\Psi_\delta = \Psi_\delta^c \cup \Psi_\delta^\alpha \cup \Psi_\delta^\beta$: Ψ_δ^c is the subset of realizations such that x_0 is inside some subdomain; Ψ_δ^α is the subset of realizations such that x_0 is at the interface between k subdomains Ω_j, $j = 1\ldots k$ for any integer k with alternating materials ε_0 and ε_1 in them and material ε_0 in Ω_1; Ψ_δ^β is the subset of realizations such that x_0 is at the interface between k subdomains Ω_j, $j = 1\ldots k$ for any integer k with alternating materials ε_1 and ε_0 in them and material ε_1 in Ω_1. Note that $\langle\nabla\varepsilon\rangle_{\Psi_\delta^c} = 0$ in the regular sense. Thus,

$$\begin{aligned}&\left|\langle\int_{B_\eta(x_0)}(\nabla\varepsilon)u\phi dx\rangle\right|\\ &\le |\varepsilon_1-\varepsilon_0|\,\|\phi\|_{L^\infty}\sum_{j=1}^{k}\|\chi_{\Omega_j}\|_{BV}\int_{G_\delta}\sum_{i=1}^{2^{N_g-1}} p^{\frac{k}{2}}(1-p)^{\frac{k}{2}}\prod_{\substack{l=1\\ l\neq j+1\ldots\\ j+k}}^{N_g} p^{1-m_l^i}\,(1-p)^{m_l^i}\,\|u_\alpha-u_\beta\|_{L^\infty}\,dG_\delta\\ &\le kKCC_p p(1-p)\|\phi\|_{L^\infty}\,|\varepsilon_1-\varepsilon_0|^2\,\delta\to 0 \quad \text{as} \quad \delta\to 0.\end{aligned}$$

Note that the inequality

$$\|u_\alpha-u_\beta\|_{L^\infty} \le kKC\,|\varepsilon_1-\varepsilon_0|\,\delta$$

comes from Lemma 2.2 and the fact that for any material in Ψ_δ^α one can find a material in Ψ_δ^β, which differs only on the subdomains Ω_j through Ω_{j+k} each with volume less than or equal to δ.

Partitions of unity argument is useful to extend local constructions to the whole domain Ω. Let us cover the compact set Ω with open balls of radius at most η. Every open cover of the compact Ω has a finite subcover, i.e. $\Omega = \cup_1^N B_\eta^i$. Now let $\{\zeta_i\}_{i=0}^\infty$ be a smooth partition of unity subordinate to the open sets B_η^i; that is suppose $0 \le \zeta_i \le 1$, $\zeta_i \in C_0^\infty(B_\eta^i)$, and $\sum_{i=0}^\infty \zeta_i = 1$ on Ω. Let $\phi \in C_0^\infty(\Omega)$. Hence, $\zeta_i\phi \in C_0^\infty(\Omega)$, $supp(\zeta_i\phi) \subset B_\eta^i$ and $\phi = \sum_i \zeta_i\phi$. Thus, the distributional derivative can be bounded

$$\begin{aligned}&\left|\left\langle\int_\Omega(\nabla\varepsilon)u\phi\,dx\right\rangle\right| \qquad (3.7)\\ &= \left|\left\langle\sum_i^N\int_{B_\eta^i}\nabla(\varepsilon)u\zeta_i\phi\ dx\right\rangle\right| \le NkKCC_p p(1-p)\|\phi\|_{L^\infty}\,|\varepsilon_1-\varepsilon_0|^2\,\delta\end{aligned}$$

Using the lower bound $|\langle u\rangle| \ge c > 0$, (3.6), and (3.7), we obtain

$$\begin{aligned}\int_\Omega|\nabla\varepsilon^*|\,dx &\le \frac{C|\varepsilon_1-\varepsilon_0|\delta\|\phi\|_{L^\infty}}{c^2} \qquad (3.8)\\ &\le C^*|\varepsilon_1-\varepsilon_0|\delta\to 0 \quad \text{as } \delta \text{ or } |\varepsilon_1-\varepsilon_0|\to 0,\end{aligned}$$

where $\nabla\varepsilon^*$ is defined in the generalized sense. Choose δ small enough that $|\langle u\rangle| \geq c > 0$. This will ensure that $\varepsilon^* \in BV(\Omega)$, and thus, we can bound the spatial variations of ε^*

$$V(\varepsilon^*, \Omega) := \sup\left\{\int_\Omega \varepsilon^* \mathrm{div}\phi \colon \phi \in C_0^1(\Omega, \mathbb{R}^n),\ \|\phi\|_{L^\infty(\Omega)} \leq 1\right\}$$
$$\leq C \int_\Omega |\nabla\varepsilon^*|\, dx \to 0 \quad \text{as } \delta \text{ or } |\varepsilon_1 - \varepsilon_0| \to 0.$$

The formula that prescribes the appropriate δ takes into account the contrast $|\varepsilon_1 - \varepsilon_0|$ in the medium (Theorem 3.1, (3.6) and (3.8)).

Note that the constant, that ε^* converges to as $\delta \to 0$, is $p\varepsilon_0 + (1-p)\varepsilon_1$ which is consistent with the quasistatic case since by letting $\delta \to 0$, we are effectively operating in the quasistatic limit.

□

4. Conclusions. When we consider wave propagation in a medium for which the size of the inhomogeneities is of the same order as the wave length, scattering effects must be accounted for and the effective dielectric coefficient is no longer a constant, but a spatially dependent function. In this paper we use novel approaches to bound the spatial variations of the effective permittivity. Related optimization problems that seek the class of materials, described by the probability density function of the geometry of the medium, that optimize certain properties of the effective permittivity will be considered in the future. An example of one such problem - the maximization of the average of the spatial variations of the effective coefficient with respect to the probability density function, is presented for one-dimensional medium in [18].

5. APPENDIX. In two dimensions using polar coordinate frame (r, θ) and assuming no incoming waves, the exterior scattered solution is

$$u_{ex}(r,\theta) = \sum_{m=1}^{\infty} A_m H_m^1(\omega r) e^{im\theta}$$

where $H_m^1(\omega r)$ are Hankel functions of first kind. Suppose that the Dirichlet datum u_{in} is given on the circle. The interior solution $u_{in} \in L^2(S_0)$, and thus it has a Fourier series representation

$$u_{in}(\theta) = \sum_{m=1}^{\infty} \hat{u}_m e^{im\theta},$$

where

$$\hat{u}_m = \frac{1}{2\pi}\int_0^{2\pi} u(\omega R_0, \theta') e^{-im\theta'}\, d\theta'.$$

The constants A_m are found from the Dirichlet condition to be

$$A_m = \frac{\hat{u}_m}{H_m^1(\omega R_0)}.$$

Thus the radiating solution is given by

$$u_s(r,\theta) = \sum_{m=1}^{\infty} \frac{H_m^1(\omega r)}{H_m^1(\omega R_0)} \hat{u}_m e^{im\theta}.$$

Differentiating in the radial direction and setting $r = R_0$ leads to

$$\frac{\partial u_s}{\partial r}(R_0,\theta) = \omega \sum_{m=1}^{\infty} \frac{\frac{\partial H_m^1}{\partial r}(\omega R_0)}{H_m^1(\omega R_0)} \hat{u}_m e^{im\theta} \equiv (Tu_s)(\theta).$$

From here we see that

$$(Tv)(\theta) = \omega \sum_{m=1}^{\infty} \left(\frac{\frac{\partial H_m^1}{\partial r}(\omega R_0)}{H_m^1(\omega R_0)} \right) \hat{v}_m e^{im\theta}, \tag{5.1}$$

where $\hat{v}_m$ are the Fourier coefficients of v, where v satisfies Helmholtz equation (2.1).

Let

$$\gamma_m \equiv \omega \frac{\frac{\partial H_m^1}{\partial r}(\omega R_0)}{H_m^1(\omega R_0)}. \tag{5.2}$$

By using the properties and identities of Hankel functions, it can be shown that $Im(\gamma_m) > 0$ and $Re(\gamma_m) < 0$ for all m.

For $m \geq 0$ and r in compact subsets of $(0, \infty)$, we have [2]

$$\left| H_m^1(\omega r) \right| \leq C \frac{2^m m!}{(\omega r)^m}.$$

The derivative of the Hankel function

$$\frac{\partial H_m^1}{\partial r}(\omega r) = \frac{m H_m^1(\omega r)}{r} - \omega H_{m+1}^1(\omega r).$$

This way we can bound the ratio

$$\left| \frac{\frac{\partial H_m^1}{\partial r}(\omega R_0)}{H_m^1(\omega R_0)} \right| \leq Cm.$$

From here we obtain the bound

$$\begin{aligned} \|Tv\|^2_{H^{-\frac{1}{2}}(S_0)} &\leq \sum_{m=1}^{\infty} (1+m^2)^{-\frac{1}{2}} \left| \frac{\frac{\partial H_m^1}{\partial r}(\omega R_0)}{H_m^1(\omega R_0)} \right|^2 |\hat{v}_m|^2 \leq \sum_{m=1}^{\infty} C(1+m^2)^{-\frac{1}{2}} m^2 |\hat{v}_m|^2 \\ &\leq \sum_{m=1}^{\infty} C(1+m^2)^{\frac{1}{2}} |\hat{v}_m|^2 \leq C\|v\|^2_{H^{\frac{1}{2}}(\Gamma_0)} \leq C\|v\|^2_{H^1(\Omega_0)}, \end{aligned} \tag{5.3}$$

where we have used the trace imbedding theorem [1].

In three dimensions using spherical coordinate frame (r, θ, ϕ) assuming $\varepsilon(x) = 1$ and no incoming waves, the scattered solution

$$u_{ex}(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} B_{lm} h_l^1(\omega r) Y_{lm}(\theta,\phi),$$

where $h_l^1(\omega r)$ are spherical Hankel functions of first kind and $Y_{lm}(\theta,\phi)$ are the normalized spherical harmonics. The latter form an orthonormal complete set of $L^2(S_0)$

[15]. Suppose that the Dirichlet datum is given on the sphere. Since $u_{in} \in L^2(S_0)$, it can be expanded into spherical harmonics as

$$u_{in}(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{u}_{lm} Y_{lm}(\theta,\phi)$$

with

$$\hat{u}_{lm} = \int_{S_0} u(R_0,\theta',\phi')\, \overline{Y_{lm}}(\theta',\phi')\, dS'.$$

The constants B_{lm} are found from the Dirichlet condition to be

$$B_{lm} = \frac{\hat{u}_{lm}}{h_l(wR_0)}.$$

Thus,

$$u_s(r,\theta,\phi) = \sum_{l=0}^{\infty} \frac{h_l^1(\omega r)}{h_l(\omega R_0)} \sum_{m=-l}^{l} \hat{u}_{lm} Y_{lm}(\theta,\phi).$$

Differentiating in the radial direction and setting $r = R_0$ gives

$$\frac{\partial u_s}{\partial r}(R_0,\theta,\phi) = \sum_{l=0}^{\infty} \omega \frac{\frac{\partial h_l^1}{\partial r}(\omega R_0)}{h_l^1(\omega R_0)} \sum_{m=-l}^{l} \hat{u}_{lm} Y_{lm}(\theta,\phi) \equiv (Tu_s)(\theta,\phi).$$

From here we see that

$$(Tv)(\theta,\phi) = \sum_{l=0}^{\infty} \omega \left(\frac{\frac{\partial h_l^1}{\partial r}(\omega R_0)}{h_l^1(\omega R_0)} \right) \sum_{m=-l}^{l} \hat{v}_{lm} Y_{lm}(\theta,\phi), \tag{5.4}$$

where $\hat{v}_{lm}$ are the coefficients in the spherical harmonics expansion of v, where v satisfies Helmholtz equation (2.1).

Let

$$\gamma_l \equiv \omega \frac{\frac{\partial h_l^1}{\partial r}(\omega R_0)}{h_l^1(\omega R_0)}. \tag{5.5}$$

The following is obtained by very slight modification of the analysis of the exterior scattering problem discussed in [10]: for all l, $Im\,\gamma_l > 0$ and $Re\,\gamma_l < 0$.

The Sobolev space $H^s(S_0)$ with real parameter s consists of all distributions f such that

$$\|f\|^2_{H^s(S_0)} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (1+\lambda_l)^s |\hat{f}_{lm}|^2 < \infty,$$

where $\hat{f}_{lm}$ are the spherical harmonics Fourier coefficients and $\lambda_l = l(l+1)$, $l \geq 0$ is the eigenvalue of the Laplace-Beltrami operator on S_0. For $l \geq 0$ and r in compact subsets of $(0,\infty)$, we have

$$\left| h_l^1(\omega r) \right| \leq C \frac{2^l l!}{(\omega r)^{l+1}}.$$

The derivative of the spherical Hankel function

$$\frac{\partial h_l^1}{\partial r}(\omega r) = \frac{1}{2}\left(\omega h_{l-1}^1(\omega r) - \frac{h_l^1(\omega r) + \omega r h_{l+1}^1(\omega r)}{r}\right).$$

This way we can bound the ratio

$$\left|\frac{\frac{\partial h_l^1}{\partial r}(\omega R_0)}{h_l^1(\omega R_0)}\right| \leq Cl.$$

From here we obtain the bound

$$\begin{aligned}\|Tv\|^2_{H^{-\frac{1}{2}}(\Gamma_0)} &\leq \omega \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (1 + l(l+1))^{-\frac{1}{2}} \left|\frac{\frac{\partial H_l^1}{\partial r}(\omega R_0)}{H_l^1(\omega R_0)}\right|^2 |\hat{v}_{l,m}|^2 \\ &\leq \omega \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C(1 + l(l+1))^{\frac{1}{2}} |\hat{v}_{l,m}|^2 \leq C\|v\|^2_{H^{\frac{1}{2}}(\Gamma_0)} \leq C\|v\|^2_{H^1(\Omega_0)},\end{aligned} \tag{5.6}$$

where we have used the trace imbedding theorem [1].

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