AN EFFECTIVE COMPLEX PERMITTIVITY FOR WAVES IN RANDOM MEDIA WITH FINITE WAVELENGTH*

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Abstract. When we consider wave propagation in random medium in the case when the wave length is comparable to feature sizes of the medium, scattering effects must be accounted for and the effective dielectric coefficient is no longer a constant, but a spatially dependent function. We obtain a bound on the spatial variations of the effective permittivity that depends on the maximum volume of the inhomogeneities and the contrast of the medium. Also a related optimization problem of maximizing the spatial average of the effective dielectric coefficient with respect to the spatial probability density function is presented. The dependence of the effective dielectric coefficient on the contrast in the medium is also investigated and an approximation formula is derived.

Key words. Random media, effective properties

AMS subject classifications. 78A48, 78A40, 78A45

1. Introduction. Usually, when one considers the propagation of an electromagnetic wave in a random medium, two length scales are of importance. The first scale is the wavelength λ of the electromagnetic wave probing the medium. The second one is the typical scale of the inhomogeneities δ. There has been plenty of work to build a theory that is applicable to wave propagation and fields that oscillate with time provided that the wavelengths associated with the fields are much larger than the microstructure. This limit where the size of the microstructure goes to zero is called the quasistatic or infinite wavelength limit. In this case the heterogeneous material is replaced by a homogeneous fictitious one whose global characteristics are good approximations of the initial ones. The solutions of a boundary value partial differential equation describing the propagation of waves converge to the solution of a limit boundary value problem which is explicitly described when the size of the heterogeneities goes to zero.

Wellander and Kristensson apply the concept of two-scale convergence, which is a well established tool in the theory of homogenization, to Maxwell equations describing the behavior of the electromagnetic fields. The method is applicable only to materials with periodic microstructure. They assume that the wavelength of the incident field is much larger than the fine scale of the composite. In this case the field cannot resolve the fine scale and the solution of the Maxwell equations can be approximated by the solution of a scattering problem with constant coefficients, i.e. the heterogeneous material has been replaced by a homogeneous material with the same effective material properties [9].

The problem of describing the effective properties of periodic materials is also investigated by Conca and Vanninatham. The approach is different since it utilizes the spectral properties of operators that represent the wave propagation through the periodic medium. But again they must assume that the period of the medium is getting smaller, so the scale of the inhomogeneities is decreasing to zero [1].

Keller and Karal [6] and Papanicolaou [8] use averaging of random realizations of materials in order to describe the effective properties of the composites when inter-

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acting with electromagnetic waves. Both analyses assume that the random materials deviate slightly from a homogeneous material, i.e. the random inclusions are very small. Papanicolaou derives formulas to compute the statistical characteristics of the reflection and transmission coefficients. Keller and Karal assume *a priori* that the effective dielectric coefficient is a constant. Using perturbation methods they approximate the dielectric constant with a complex number, whose imaginary part accounts for the wave attenuation.

A great overview of the subject of wave propagation in random media is given in a book by Ishimaru [5]. Also recent results in this field could be found in the AMS-IMS-SIAM proceedings edited by Kuchment [7].

All of the above methods that provide bounds and describe the behavior of the dielectric coefficients do not account for scattering effects which occur when the wavelength is no longer much larger than the inhomogeneities of the composite. This problem has remained open and results are sparse. The problem is extremely difficult and none of the techniques that come from the quasistatic regime can be applied directly to the scattering problem since all of the quasistatic methods utilize the condition that the size of heterogeneities goes to zero.

Wave localization and cancellation must be accounted for when the wavelength is in the same order as the size of the heterogeneities, which means that the effective coefficients are no longer constants as in the quasistatic case, but functions of the space variable.

Even the correct definition of "effective medium" is somewhat unclear outside the quasistatic regime. In this work, we assume that the purpose of the effective medium is to reproduce the average or expected wave field as the actual medium varies over a given set of random realizations. The definition of the effective dielectric coefficient as the ratio of the average of the media times the field vector over all material realizations and the average of the field vector over all material realization, i.e. $\epsilon^* = \frac{\langle \epsilon u \rangle}{\langle u \rangle}$, comes naturally from the equations. It does not prevent spatial variations and is consistent with the definition of the effective dielectric constant in the quasistatic regime. There

$$\epsilon^* \langle E \rangle = \langle D \rangle = \langle \epsilon E \rangle,$$

where the averaged electric field $\langle E \rangle = \bar{E}$, a given constant, and the averaged dielectric displacement $\langle D \rangle = \int_{\Psi_\delta} D \, dP = \int_{\Psi_\delta} \epsilon(x, \psi) E(x, \psi) \, dP$ is independent of x which ensures that ϵ^* in the quasistatic case is a constant.

2. Model problem. In this paper we consider time-harmonic electromagnetic wave propagation through nonmagnetic ($\mu = 1$) heterogeneous two-component layered media. The geometry of the medium is obtained by dividing the interval (0, 1) in subintervals whose length is less than or equal to δ and assigning material of type one with probability $1 - p$ or material of type two with probability p to each subinterval. The wave propagation is perpendicular to the layers. Assuming that the electric field vector $E = (0, 0, u)$, Maxwell's equations reduce to the Helmholtz equation. Let χ be an indicator function. Choosing the support of χ to be bounded inside the interval (0, 1), and enforcing an outgoing wave condition leads to the two-point boundary problem

$$u'' + \omega^2 \epsilon u = f, \quad \text{in } \Omega \equiv (0, 1), \tag{2.1}$$

$$u'(0) + i\omega u(0) = 0, \tag{2.2}$$

$$u'(1) - i\omega u(1) = 0, \tag{2.3}$$

where ω represents the frequency, and $\epsilon(x,\psi) = 1 + z\chi(x,\psi)$ is the random dielectric coefficient that can take values of $\epsilon(x) = 1$ and $\epsilon(x) = 1 + z$ in each layer for a given realization ψ. The boundary conditions come from the continuity of the solution u and u' on the interface between the homogeneous medium (where $u = e^{i\omega x}$) and the layered medium.

3. Probability space and definition of effective dielectric coefficient. The spatial domain Ω is partitioned into N_g disjoint intervals $\{\Omega_j\}_{j=1}^{N_g}$, and the material in each subinterval is assigned. Here g is the geometry, and it accounts for possible differences in length of the intervals. This gives us one realization. The set of realizations Ψ_δ is infinite in general since the set of all geometries, Γ_δ, can be infinite. Fix a geometry g. Denote the set of realizations for geometry g by R_g:

$$R_g = \{m_g = (m_1, \ldots, m_{N_g}) : m_j = 0 \text{ or } m_j = 1,\, j = 1, \ldots, N_g\}$$

The set R_g has 2^{N_g} elements. Thus the set of material realizations, Ψ_δ is described as follows,

$$\Psi_\delta = \{(g, m_g) : g \in \Gamma_\delta,\, m_g \in R_g\}.$$

The probability measure is

$$P = \sum_{m_g \in R_g} \prod_{j=1}^{N_g} p^{1-m_j}\,(1-p)^{m_j}\, G_\delta,$$

where G_δ is the probability measure on the space of all geometries, Γ_δ. The product describes the multiplication of the probabilities of the materials in each subdomain Ω_j, which is summed over the set of all realizations for a particular geometry g. We can find the expected value of the solution u of the Helmholtz equation (2.1), that depends on the random variables through its dependence on the composite material:

$$\langle u \rangle = \int_{\Psi_\delta} u\, dP \tag{3.1}$$

where $\langle\rangle$ denotes expected value - an average over the set of all random realizations. We average over all the possible realizations to obtain the equation

$$\langle u \rangle'' + \omega^2 \langle \epsilon u \rangle = f,$$

where the interchange of the derivative and integral is possible due to the uniform bound of all solutions u. We seek to find the dielectric coefficient that will solve the problem

$$\langle u \rangle'' + \omega^2 \epsilon^* \langle u \rangle = f, \tag{3.2}$$

where $\langle u \rangle$ is the averaged solution. From the above two equation, it is easy to see that the appropriate definition for ϵ^* is

$$\epsilon^* = \frac{\langle \epsilon u \rangle}{\langle u \rangle}. \tag{3.3}$$

Note that the definition of ϵ^* does not preclude spatial variations $\epsilon^* = \epsilon^*(x)$.

We will illustrate in this paper that as ω increases (which will decrease the wavelength), we begin to see spatial variations in the effective dielectric coefficient due to the presence of scattering effects, and it is no longer a constant. But nevertheless ϵ^* as defined in (3.3) is the "correct" definition of the effective dielectric coefficient, in that it reproduces the average field response through equation (3.2).

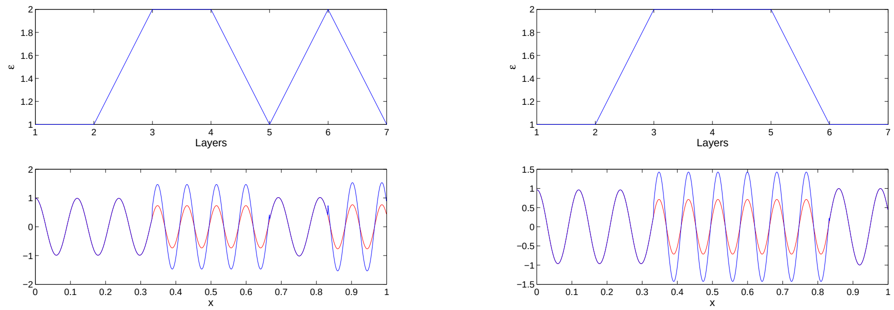

FIG. 4.1. *Sample realizations in a six layers medium. Left:* ϵ *and real part of* ϵu *(blue) and* u *(red); right:* ϵ *and real part of* ϵu *(blue) and* u *(red).*

4. Spatial dependence of the effective dielectric coefficient. We observe the spatial dependence of the effective dielectric coefficient by numerically calculating ϵ^* and graphing it as a function of x. In these numerical experiments the ϵ^* is calculated by dividing the interval $(0, 1)$ into the corresponding number of intervals m, each layer of length $\frac{1}{m}$ and going through all possible realizations by assigning in each layer either material of type one or material of type two, both with probability $\frac{1}{2}$. The solution u for each particular layered material is computed by the transport matrix method. Sample realizations in the case of a six-layer medium are given in Figure 4.1. In these numerical experiments $z = 1$ and $\omega = 53$. The graph on the left shows the solution u and the product ϵu in a medium composed of material of type one ($\epsilon_1 = 1$) in the first, second, and fifth layers, and material of type two ($\epsilon_2 = 2$) in the third, forth, and sixth layers. The graph on the right shows the solution u and the product ϵu in a medium composed of material of type one ($\epsilon_1 = 1$) in the first, second, and sixth layers, and material of type two ($\epsilon_2 = 2$) in the third, forth, and fifth layers. The expected $\langle u\rangle$ is obtained by evaluation the solution u for each realization and multiplying it by the probability of the particular realization, i.e.

$$\langle u\rangle = \sum_{m_g\in R_g} u(x, m_g) \prod_{j=1}^{N_g} p^{1-m_j}\,(1-p)^{m_j}\,.$$

In the case when both materials are assigned according to probability $\frac{1}{2}$, each solution u is multiplied by $\left(\frac{1}{2}\right)^m$. The expected $\langle \epsilon u\rangle$ is computed similarly. We observe that when the length of the layers is $1/6$ the spatial variations of ϵ^* are more pronounced than in the case when the length of the layer is $1/16$ (Figure 4.2).

Numerical experiments also show that the spatial variations decrease in magnitude when the contrast z between the two materials is small (Figure 4.3). In these experiments we are looking at a sixteen layers medium and $\omega = 53$. We vary the contrast. In the first experiment, we assign material of type one ($\epsilon_1 = 1$) or material of type two ($\epsilon_2 = 1.5$), both with probability $\frac{1}{2}$. In the second experiment, we assign material of type one ($\epsilon_1 = 1$) or material of type two ($\epsilon_2 = 13$), both with probability $\frac{1}{2}$. The dependence of the magnitude of the spatial variations on the contrast in the medium is obvious.

The spatial variations decrease in magnitude as the length of the layers decreases (which increases the number of layers in Ω) and also as z decreases. A bound on

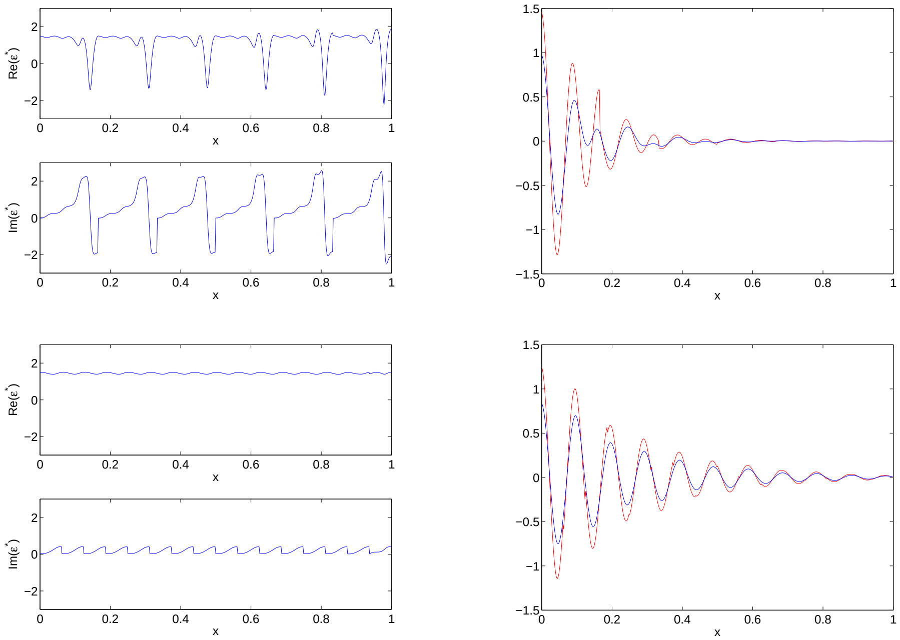

FIG. 4.2. *Spatial Variations. Upper left: Real and imaginary ϵ^* in a medium of six layers; upper right: Real part of $\langle \epsilon u \rangle$ (red) and $\langle u \rangle$ (blue) in a medium of six layers; lower left: Real and imaginary ϵ^* in a medium of sixteen layers; lower right: Real part of $\langle \epsilon u \rangle$ (red) and $\langle u \rangle$ (blue) in a medium of sixteen layers.*

the spatial variations that depends on the maximum length of the layers and on the contrast and a uniform upper bound on the effective dielectric coefficient is derived analytically. The proof shows that as the length of the layers goes to zero, we have proved that ϵ^* converges to a constant consistent with the one obtained through homogenization ($\epsilon^* \to 1 + zp$ as $\delta \to 0$). The proof is rather technical and will be presented in the subsequent paper where it is extended to two- and three-dimensional random medium [2]. The proof utilizes the fact that at a particular x we can represent $\langle u \rangle$ as the sum of realizations that have material of type one at that particular x, $\langle u \rangle_{\Psi_1}$, and realizations that have material of type two at that x, $\langle u \rangle_{\Psi_2}$. To each realization in $\langle u \rangle_{\Psi_1}$ corresponds a realization in $\langle u \rangle_{\Psi_2}$ such that the material ϵ differs only in the layer where x is. As the maximum size of the layers δ or as the contrast z decreases, the two solutions, $u_1 \in \langle u \rangle_{\Psi_1}$ and $u_2 \in \langle u \rangle_{\Psi_2}$ for the two similar media become closer, and their difference can be bounded, where the bound depends on both δ and z. This makes the conditional expectations $\langle u \rangle_{\Psi_1}$ and $\langle u \rangle_{\Psi_2}$ and their spatial derivatives very close when the size of the layers δ or z are small. Also when either of the above conditions is satisfied, the probability that the magnitude of solutions u will be within a certain radius α from the magnitude of the solution of the constant boundary value problem with dielectric constant $1 + zp$ goes to one as δ or $z \to 0$. Thus by picking the radius sufficiently small, we can ensure that we can bound $|\langle u \rangle|$ from below by a constant c. These facts allow us to prove that in the distributional sense $\frac{d\epsilon^*}{dx} \to 0$ as $\delta \to 0$ or as $z \to 0$. We have proved that the constant, that ϵ^* converges to, is $1 + zp$ which is consistent with the quasistatic regime since by letting $\delta \to 0$, we are effectively operating in the quasistatic limit.

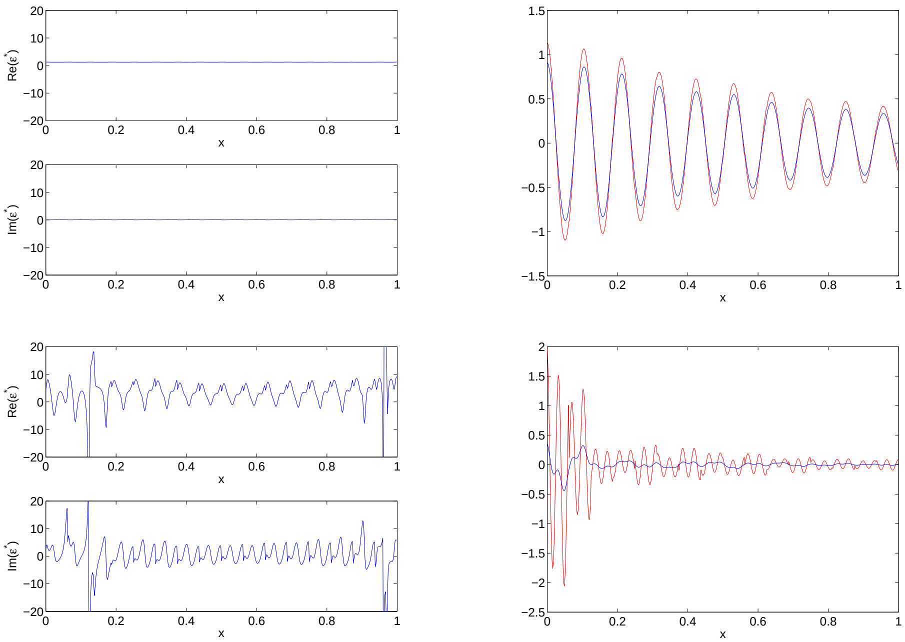

FIG. 4.3. *Spatial Variations. Upper left: Real and imaginary ϵ^* in a medium with contrast $z = 0.5$; upper right: Real part of $\langle \epsilon u \rangle$ (red) and $\langle u \rangle$ (blue) in a medium of sixteen layers and contrast $z = 0.5$; lower left: Real and imaginary ϵ^* in a medium of sixteen layers and contrast $z = 12$; lower right: Real part of $\langle \epsilon u \rangle$ (red) and $\langle u \rangle$ (blue) in a medium of sixteen layers and contrast $z = 12$.*

The proof also shows that if we choose a maximum length of the inhomogeneities δ to be such that $|\langle u \rangle| \geq c$, we can bound the spatial variations of ϵ^*. The formula that prescribes the appropriate δ takes into account the frequency of the incident field ω and the contrast z in the medium [2].

5. Approximation formulas. Let g_ω be the free-space Green's function for the operator $Lv = v'' + \omega^2 v$ (with the outgoing wave condition). Our problem can be rewritten to yield the Lippmann-Schwinger equation

$$u(x) = -z\omega^2 \int_\Omega g_\omega(x-y)\chi(y)u(y)dy + q(x), \tag{5.1}$$

where $q = g_\omega \star f$. Define the operator $A_\omega : L^2(\Omega) \to L^2(\Omega)$ by

$$(A_\omega v)(x) = \int_\Omega g_\omega(x-y)v(y)dy, \quad x \in \Omega. \tag{5.2}$$

In the case when $|z\omega^2| \|A_\omega\| < 1$,

$$u = (I + z\omega^2 A_\omega \chi)^{-1} q, \tag{5.3}$$

and the Neumann series

$$u = q - z\omega^2 A_\omega \chi q + z^2\omega^4 (A_\omega \chi)^2 q - \dots \tag{5.4}$$

converges absolutely. Thus, the effective dielectric coefficient can be represented as

$$\epsilon^* = 1 + z\frac{pq - z\omega^2\langle\chi A_\omega\chi\rangle q + z^2\omega^4\langle\chi A_\omega\chi A_\omega\chi\rangle q - \dots}{q - zp\omega^2 A_\omega q + z^2\omega^4 A_\omega\langle\chi A_\omega\chi\rangle q - \dots}. \tag{5.5}$$

In the case of small z every term in the series is a constant provided the medium is stationary. In such media, the correlation functions depend only on the distance between the points, and not their positions. In this case all the correlation functions depend on the distance between all of the points, i.e. the three-point correlation function $N(x,y,s) = N(|x-y|,|x-s|,|s-y|)$. An example of such medium is one constructed by varying the length of the first layer (and thus the last to compensate), and leaving the length of the middle layers constant. The material in each layer is assigned with a chosen probability. In order to do that we must develop a method to evaluate the terms of form $\langle\chi A_\omega\chi\rangle q$, $\langle\chi A_\omega\chi A_\omega\chi\rangle q$, etc. Keller and Karal suggest how this can be done when calculating $\langle\chi A_\omega\chi\rangle q$. The mean value theorem for the solution $q = g_\omega \star f$ of the constant coefficient problem

$$u'' + \omega^2 u = f$$

is applied [6], where f is the delta function and q is a plane wave solution. The Green's function for this problem is

$$g_\omega = \frac{ie^{i\omega|x|}}{2\omega}.$$

Knowing the correlation function $N(x,y) = N(|x-y|) = N(r)$ and using the mean value theorem in one dimension, we can calculate

$$\begin{aligned}\langle\chi A_\omega\chi\rangle q &= (p-p^2)\int g_\omega(|x-y|)N(|x-y|)q(y)\,dy + p^2\int g_\omega(|x-y|)q(y)\,dy \\ &= 2(p-p^2)\int_0^\infty g_\omega(r)N(r)cos(\omega r)\,dr\;q(x) + 2p^2\int_0^\infty g_\omega(r)cos(\omega r)\,dr\;q(x)\end{aligned}$$

In a medium where the length of the first layer varies from 0 to d, the correlation function is

$$N(x,y) = N(|x-y|) = \begin{cases} 1 - \frac{|x-y|}{d} & \text{when } 0 \le |x-y| \le d; \\ 0 & \text{otherwise.}\end{cases}$$

Calculating the integrals for $r \in supp(N)$, we obtain

$$\langle\chi A_\omega\chi\rangle q = \left(-\frac{(p-p^2)i}{8\omega^3 d}(2d^2\omega^2 - 1 + e^{2i\omega d}) + \frac{p}{4\omega^2}(pe^{2i\omega d} + 2i\omega d - 1)\right)q(x).$$

The mean value theorem is applied repeatedly to calculate the multiple integrals of the form $\langle\chi A_\omega\dots\chi A_\omega\chi\rangle q$, when the N-point correlation is given. Once these are calculated our formula (5.5) gives the approximation to the needed order, e.g.

$$\varepsilon^* \approx 1 + z\frac{p - z\omega^2\left(-\frac{(p-p^2)i}{8\omega^3 d}(2d^2\omega^2 - 1 + e^{2i\omega d}) + \frac{p}{4\omega^2}(pe^{2i\omega d} + 2i\omega d - 1)\right)}{1 - zp\left(\frac{id}{2\omega} + \frac{e^{2i\omega d}}{4\omega^2} + \frac{1}{4\omega^2}\right) + z^2\omega^4\left(\frac{id}{2\omega} + \frac{e^{2i\omega d}}{4\omega^2} + \frac{1}{4\omega^2}\right)\left(-\frac{(p-p^2)i}{8\omega^3 d}(2d^2\omega^2 - 1 + e^{2i\omega d}) + \frac{p}{4\omega^2}(pe^{2i\omega d} + 2i\omega d - 1)\right)}.$$

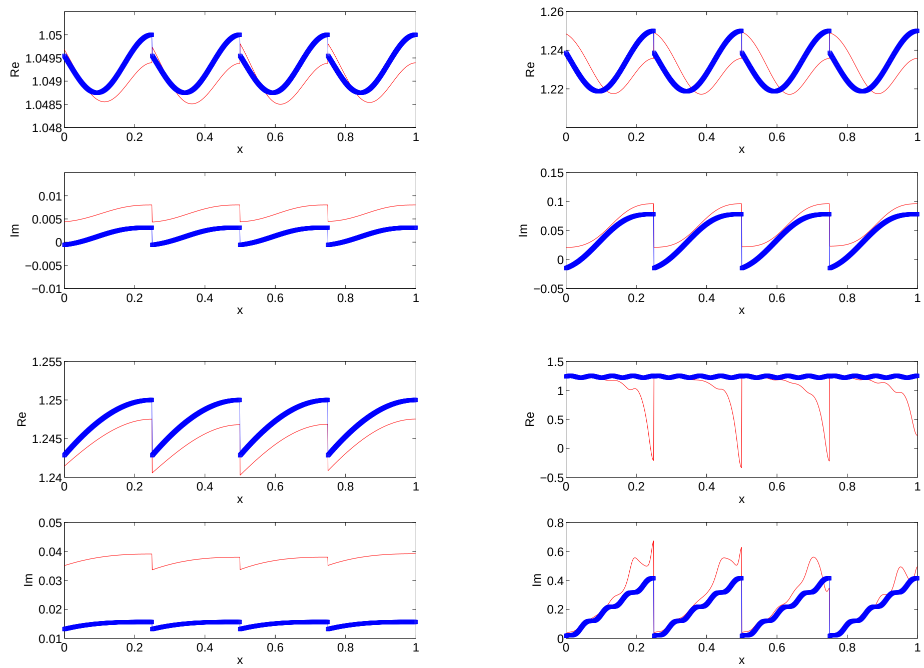

FIG. 5.1. *Second Order Spatially Dependent Approximation.*

Numerical experiments show that in a media with a correlation function depending on position, the best approximation may be a function of the space variable. In the numerical experiments illustrated in Figure 5.1 we have graphed the real and imaginary parts of ϵ^* and its second order spatially dependent approximation, calculated using (5.5). The appropriate correlation function for the medium is spatially dependent and assigns 1 (or fully correlated), if the two points are in the same interval and 0 (no correlation), otherwise. Since the expansion is done around $z = 0$, it gives better approximation for small z's and ω's. In the experiment, depicted in the upper left of Figure 5.1, we use a medium of four layers and contrast $z = 0.1$, when the frequency $\omega = 10$. We see that our second order approximation (blue) gives a very good approximation of both the real and imaginary parts of ϵ^* (red), capturing the spatial variations. In the upper right of Figure 5.1, we have displayed real and imaginary ϵ^* (red), and its second order spatially dependent approximation (blue) in a medium of four layers, contrast $z = 0.5$, when the frequency $\omega = 10$. In the lower left of Figure 5.1, we observe the real and imaginary parts of ϵ^* (red), and its second order spatially dependent approximation (blue) in a medium of four layers, contrast $z = 0.5$, and $\omega = 2$. The approximation is very good in the case when z and ω are small even if we have only four, relatively long, layers. In the numerical experiment depicted in the lower right of Figure 5.1, $\omega = 53$. For large frequencies, we expect the approximation to fail, but nevertheless, we see that our second order spatially dependent approximation (blue) captures some of the behavior of the real and imaginary parts of ϵ^* (red). In this experiment, we are looking at a medium of four layers and contrast $z = 0.5$.

6. Optimization of the effective dielectric constant. Another interesting and pertinent question that one may ask when talking about effective properties of materials is finding the class of materials that optimizes the effective coefficient. We look at one such question, that looks for the class of materials that maximizes the spatial average of the effective coefficient. Posed in this way, the problem is global. But many other problems, optimizing both local and global properties of the effective coefficient, can be answered using similar techniques.

Denote by h the probability density function with respect to the geometry; h will depend on the length of each layer and these lengths will be random variables: $h(d_1, d_2, ..., d_n)$. Thus, the effective dielectric coefficient will be calculated by

$$\epsilon^* = \frac{\int_{\Psi_\delta} \epsilon u h \, dP}{\int_{\Psi_\delta} u h \, dP}$$

.

We can formulate an optimization problem that will look for the probability density function of the geometry h_0 that will maximize the spatial average of the effective dielectric coefficient:

$$\max_h \int_\Omega \epsilon^* dx$$

or equivalently,

$$\max_h \int_\Omega \frac{\int_{\Psi_\delta} \epsilon u h \, dP}{\int_{\Psi_\delta} u h \, dP} dx \qquad \text{subject to}$$

$$\int_{\Psi_\delta} h \, dP = 1;$$

$$h(\mathbf{d}) \geq 0 \quad \text{for all } \mathbf{d};$$

$$\sum_i d_i = |\Omega|.$$

The effective dielectric coefficient ϵ^* is a continuous function of h, provided the lengths d_i are sufficiently small. Continuity ensures the existence and uniqueness of a maximizing probability density function h_0.

In the numerical experiments we divide $\Omega = [0, 1]$ into three subintervals, each of length $d_1, d_2, d_3 = 1 - d_1 - d_2$. Both d_1 and d_2 are pulled from the array of possible lengths $[0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.9]$, and $d_3 = 1 - d_1 - d_2$ provided it is positive. We solve the scattering problem for each material (each realization) and look for material geometries (characterized by d_1 and d_2) that will give us the maximum spatial average of ϵ^* and weigh them appropriately by h_0. We have solved the optimization problem for two different frequencies and have displayed the results in Figure 6.1.

7. Conclusions. When we consider wave propagation in one dimensional layered medium in the case when the size of the inhomogeneities is of the same order as the wave length, scattering effects must be accounted for and the effective dielectric coefficient is no longer a constant, but a spatially dependent function. We are interested in bounding the spatial variations and the bound depends on the maximum length of the inhomogeneities, the frequency of the wave, and the contrast of the medium. The proofs of the main theorem are novel and will be described in a

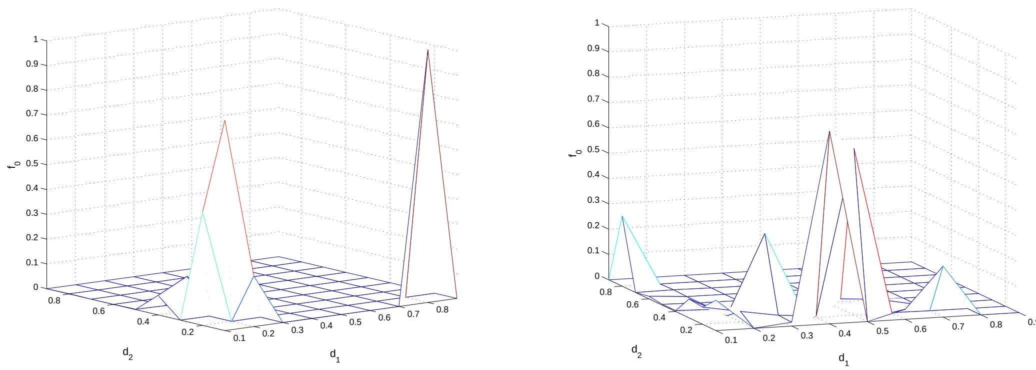

FIG. 6.1. *Maximizing probability density function. Left: Maximizing probability density function for a medium of three layers, contrast* $z = 1$*,* $\omega = 10$*, maximum* $\int_\Omega \epsilon^* dx = 1.437$*; Right: Maximizing probability density function for a medium of three layers, contrast* $z = 1$*,* $\omega = 100$*, maximum* $\int_\Omega \epsilon^* dx = 1.0692$*.*

subsequent paper [2]. The proof considers one-, two-, and three-dimensional random medium. Also a related optimization problem of maximizing the spatial average of the effective dielectric coefficient with respect to the spatial probability function is investigated and numerical results are presented.

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