The characteristic polynomial of an algebra and representations

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Suppose that \( \mathbf{k} \) is a field and let \( A \) be a finite dimensional, associative, unital \( \mathbf{k} \)-algebra. Often one is interested in studying the finite-dimensional representations of \( A \). Of course, a finite dimensional representation of \( A \) is simply a finite dimensional \( \mathbf{k} \)-vector space \( M \) and a \( \mathbf{k} \)-algebra homomorphism \( A \to \text{End}_\mathbf{k}(M) \). In this article we will not consider representations of algebras, but rather how to determine if a \( \mathbf{k} \)-linear map \( \phi : A \to \text{End}_\mathbf{k}(M) \) is actually a homomorphism. We restrict our attention to the case where \( A \) is a product of field extensions of \( \mathbf{k} \). If \( \phi : A \to \text{End}_\mathbf{k}(M) \) is a representation then certainly, if \( a \in A \) satisfies \( a^n = 1 \) then \( \phi(a)^m = \text{id} \) as well. Our first Theorem is a remarkable converse to this elementary observation.

**Theorem A.** Suppose that \( A \) is a product of field extensions of \( \mathbf{k} \) and \( \phi : A \to \text{End}_\mathbf{k}(M) \) is a \( \mathbf{k} \)-linear map. Let \( n > 2 \) be a natural number and assume that \( \mathbf{k} \) has \( n \) primitive \( n \)-th roots of unity. If \( \phi(1_A) = \text{id}_M \) and for each \( a \in A \) such that \( a^n = 1 \), \( \phi(x)^n = \text{id} \) then \( \phi \) is an algebra homomorphism.

Consider the regular representation \( \mu_L : A \to \text{End}_\mathbf{k}(A) \) of \( A \) on itself by left multiplication. For \( a \in A \), let \( \chi_a(t) \) and \( \mu_a(t) \) be the characteristic and minimal polynomials of \( \mu_L(a) \), respectively. We note that \( \chi_a(a) = \mu_a(a) = 0 \) in \( A \). Therefore if \( M \) is a finite dimensional left \( A \)-module with structure map \( \phi : A \to \text{End}_\mathbf{k}(M) \) then \( \chi_a(\phi(a)) = \mu_a(\phi(a)) = 0 \) in \( \text{End}_\mathbf{k}(M) \). The notion of assigning a characteristic polynomial to each element of an algebra and considering representations which are compatible with this assignment has appeared in [Pro87]. This idea has been applied to some problems in noncommutative geometry as well [LB03]. However, as far as we know the following related notion is new.

**Definition 1.** Suppose that \( \phi : A \to \text{End}_\mathbf{k}(M) \) is a \( \mathbf{k} \)-linear map, where \( M \) is a finite dimensional \( \mathbf{k} \)-vector space. We say that \( \phi \) is a characteristic morphism if \( \chi_a(\phi(a)) = 0 \) for all \( a \in A \). We say that \( \phi \) is minimal-characteristic if, moreover, \( \mu_a(\phi(a)) = 0 \) for all \( a \in A \).

It is natural to ask whether or not the notions of characteristic morphism and minimal characteristic morphism are weaker than the notion of algebra morphism. Let us address minimal-characteristic morphisms first.

**Corollary.** Assume that \( A \) is a product of \( d > 2 \) field extensions of \( \mathbf{k} \) and that \( \mathbf{k} \) has a full set of \( d \)-th roots of unity. Then a minimal-characteristic morphism \( \phi : A \to \text{End}_\mathbf{k}(M) \) is an algebra morphism.

**Proof.** First note that \( \mu_1(t) = t - 1 \). Hence \( \phi(1) = \text{id} \). Furthermore if \( a \in A \) satisfies \( a^d = 1 \) then \( \mu_a(t) \) divides \( t^d - 1 \). Therefore, \( \phi(a)^d = \text{id} \). Hence, Theorem A implies that \( \phi \) is an algebra morphism.

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**Example 2.** Let \( a, b \in k \) be such that \( a + b \neq 0 \). Then the map \( \phi : k^{x2} \to \text{Mat}_2(k) \) given by

\[
\phi(e_1) = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}, \quad \phi(e_2) = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}
\]

is a characteristic morphism that is not a representation.

Characteristic morphisms form a category in a natural way. Any linear map \( \phi : A \to \text{End}_k(M) \) endows \( M \) with the structure of a \( T(A) \) module, where \( T(A) \) denotes the tensor algebra on \( A \). We can view the characteristic polynomial of elements of \( A \) as a homogeneous form \( \chi(t) \in \text{Sym}^*(A^\vee)[t] \) of degree \( d \), monic in \( t \). Pappacena [Pap00] associates to such a form an algebra

\[
C(A) = \frac{T(A)}{(\chi_a(a) : a \in A)},
\]

where if \( \chi_a(t) = \sum_{i=0}^{d} c_i(a)t^i \) then

\[
\chi_a(a) := \sum_{i=0}^{d} c_i(a)a^\otimes_i \in T(A).
\]

Clearly, the action map \( \phi : A \to \text{End}_k(M) \) of a \( T(A) \)-module \( M \) is a characteristic morphism if and only if the action of \( T(A) \) factors through \( C(A) \). We declare the category of characteristic morphisms to be the category of finite-dimensional \( C(A) \)-modules. So we have a notion of irreducible characteristic morphism. The characteristic morphism constructed in Example 2 is not irreducible, being an extension of two irreducible characteristic morphisms. However, every irreducible characteristic morphism \( k^{x2} \to \text{End}_k(M) \) is actually an algebra morphism. On the other hand, this is not always the case.

**Example 3.** The linear map \( k^{x3} \to \text{Mat}_3(k) \) defined by

\[
e_1 \mapsto \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad e_2 \mapsto \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}, \quad e_3 \mapsto \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}
\]

is an irreducible characteristic morphism.

Given a linear map \( \phi : A \to \text{End}_k(M) \), let \( T_\phi \in A^\vee \otimes \text{End}_k(M) \) be the element that corresponds to \( \phi \) under the isomorphism \( \text{Hom}_k(A, \text{End}_k(M)) \cong A^\vee \otimes \text{End}_k(M) \). We view \( T_\phi \) as an element of \( \text{Sym}^*(A^\vee) \otimes \text{End}_k(M) \). The equation \( \chi(a)(\phi(a)) = 0 \) for all \( a \in A \) holds if and only if \( \chi_A(T_\phi) = 0 \) in \( \text{Sym}^*(A^\vee) \otimes \text{End}_k(M) \). We can just as easily view \( T_\phi \) as an element of \( T(A^\vee) \otimes \text{End}_k(M) \). Moreover, we can lift \( \chi \) from \( \text{Sym}^*(A^\vee)[t] \) to \( T(A^\vee) \otimes k[t] \) by the naïve symmetrization map \( \text{Sym}^*(A^\vee)[t] \to T(A^\vee) \otimes k[t] \).

**Theorem B.** Assume that \( \text{char}(k) = 0 \) or greater than \( d \). Let \( A = k^{x\cdot d} \) and \( \phi : A \to \text{End}_k(M) \) a \( k \)-linear map. The map \( \phi \) factors through a homomorphism \( A \to \text{End}_k(M) \) if and only if \( \chi(T) = 0 \) in \( T(A^\vee) \otimes_k B \).

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Proofs

We now turn to the proofs of the results in the introduction. The proof of the Theorem 4A depends on an arithmetic Lemma.

Lemma 4. Let \( \zeta \in k \) be a primitive \( n \)th root of unity. Suppose that \( a, b, c, d \in \mathbb{Z} \) satisfy \( b, d \neq 0 \mod n \) and
\[
\frac{\zeta^a - 1}{\zeta^b - 1} = \frac{\zeta^c - 1}{\zeta^d - 1}.
\]
Then either:

1. \( a \equiv b \mod n \) and \( c \equiv d \mod n \), or
2. \( a \equiv c \mod n \) and \( b \equiv d \mod n \).

Proof. After possibly passing to a finite extension we may assume that \( k \) admits an automorphism sending \( \zeta \) to \( \zeta^{-1} \). Thus we have
\[
\frac{\zeta^{-a} - 1}{\zeta^{-b} - 1} = \frac{\zeta^{-c} - 1}{\zeta^{-d} - 1},
\]
which we rewrite
\[
\frac{\zeta^{-a} - \zeta^{-c}}{\zeta^{-b} - \zeta^{-d}} = \frac{\zeta^{-e} - 1}{\zeta^{-d} - 1}.
\]
Using our assumption we find that \( \zeta^{-a} = \zeta^{-c} \). Thus \( b - a \equiv d - c \mod n \). Let \( e = b - a \equiv d - c \mod n \). Then we have
\[
\frac{\zeta^{b-e} - 1}{\zeta^b - 1} = \frac{\zeta^{d-e} - 1}{\zeta^d - 1},
\]
which implies that
\[
\zeta^{b-e} + \zeta^d = \zeta^{d-e} + \zeta^b.
\]
Finally we see that
\[
\zeta^d - \zeta^b = (\zeta^d - \zeta^b)\zeta^{-e}
\]
Therefore either \( e \equiv 0 \mod d \) so that (1) holds, or \( d \equiv b \mod d \) so that (2) holds. \( \square \)

Proof of Theorem 4A. Whether or not \( \phi \) is an algebra homomorphism is stable under passage to the algebraic closure of \( k \). So we may assume that \( k \) is algebraically closed, and identify \( A \cong k^d \) for \( d = \text{dim}_{k}(A) \). Let \( e_1, \ldots, e_d \in A \) be a complete set of orthogonal idempotents. Put \( \alpha_i = \phi(e_i) \) and note that by hypothesis \( \alpha_1 + \cdots + \alpha_d = \text{id} \). Fix a primitive \( n \)th root of unity \( \xi \). Then \( x = 1 + (\xi - 1)e_i \) satisfies \( x^n = 1 \). Therefore \( \phi(x)^d = \text{id} \). This implies that \( \phi(x) \) is diagonalizable and each eigenvalue is an \( n \)th root of unity. Now, since \( \phi \) is linear,
\[
\alpha_i = \frac{\phi(x) - \text{id}}{\xi - 1}
\]
and hence \( \alpha_i \) is diagonalizable as well. Let \( \lambda \) be an eigenvalue of \( \alpha_i \). Then for some \( a \) we have
\[
\lambda = \frac{\xi^a - 1}{\xi - 1}.
\]
Now for any \( b \), \( \phi(1 + (\xi^b - 1)e_i)^d = \text{id} \). So we see that
\[
1 + \lambda(\xi^b - 1)
\]
must be a root of unity for every $b$. However, if
\[ 1 + \lambda(\xi^b - 1) = \xi^c \]
then Lemma 4 implies that either $a \equiv 1 \mod n$, $\lambda = 0$, or $b \equiv 1 \mod n$. Now, $b$ is under our control and since $n \geq 3$ we can choose $b \neq 0, 1 \mod n$, excluding the third case. If $a \equiv 1 \mod n$ then $\lambda = 1$ and otherwise $\lambda = 0$. Thus $\alpha_i$ is semisimple with eigenvalues equal to zero or one. So $\alpha_i^2 = \alpha_i$.

Let $i \neq j$ and consider
\[ y_a = \text{id} + (\xi^a - 1)(\alpha_i + \alpha_j) \]
Clearly, $y_a^n = \text{id}$ and thus $y_a$ is semisimple. We compute
\[ (y_a - \text{id})^2 = (\xi^a - 1)^2(\alpha_i\alpha_j + \alpha_j\alpha_i) + (\xi^a - 1)(y_a - \text{id}) \]
and deduce that
\[ (\xi^a - 1)^{-2}(y_a - \text{id})(y_a - \xi^c) = (\alpha_i\alpha_j + \alpha_j\alpha_i). \] (1)

Assume that $b \neq 0 \mod n$. Observe that $y_a - \text{id} = \xi^{a-1}(y_b - \text{id})$ and therefore, $y_a$ and $y_b$ are simultaneously diagonalizable. Suppose that $\xi^c$ is an eigenvalue of $y_b$. Then
\[ \frac{\xi^a-1}{\xi^b-1} = \xi^c \]
is an eigenvalue of $y_a$. Since $n \geq 3$ we can assume that $a \neq b, 0 \mod n$. Then Lemma 4 implies that $c \equiv a \mod n$ and $b \equiv c \mod n$. Since $\xi^c$ was any eigenvalue of $y_b$ we find that $y_b = \xi^b\text{id}$. This means that the right side of (1) vanishes. So $\alpha_i\alpha_j = -\alpha_j\alpha_i$ for all $i, j$. Suppose that $\alpha_i(m) = m$. Then $\alpha_j(\alpha_i(m)) = \alpha_j(m) = -\alpha_i(\alpha_j(m))$. Since $1$ is not an eigenvalue of $\alpha_i$ we see that $\alpha_j(m) = 0$. Now let $m \in M$ and write $m = m_0 + m_1$ where $\alpha_i(m_0) = 0$ and $\alpha_i(m_1) = m_1$. Then
\[ \alpha_i(\alpha_j(m)) = \alpha_i(\alpha_j(m_0)) = -\alpha_j(\alpha_i(m_0)) = 0. \]
Thus we see that in fact $\alpha_i\alpha_j = 0$. So $\alpha_1, \ldots, \alpha_d$ satisfy the defining relations of $k^{\times d}$ and $\phi$ is actually an algebra homomorphism.

We now turn to the proof of Theorem B. The key idea is to use the fact that the single equation $\chi(T_a) = 0$ over the tensor algebra encodes many relations for the matrices defining $\phi$. It is convenient to consider $X_i = \phi(e_i)$, where $e_i$ is the standard basis of idempotents in $k^{\times d}$. Furthermore we write $\chi_d$ for the characteristic polynomial of $k^{\times d}$ viewed as an element of $k(x_1, \ldots, x_d, t)$ (where $x_1, \ldots, x_d$ is the dual basis to $e_1, \ldots, e_d$).

**Lemma 5.** Suppose that $k$ is a field with char($k$) $> d$. Let $X_1, \ldots, X_d \in M_n(k)$ and put $T = x_1X_1 + \ldots + x_dX_d$. If $T$ satisfies $\chi_d$ then

1. for some $i = 1, \ldots, d$, $X_i$ has a 1-eigenvector, and
2. if $m \in k^n$ satisfies $X_im = m$ then $X_jm = 0$ for all $j \neq i$.

**Proof.** (1.) Let $S = k[x_1, \ldots, x_d]$ as an $A = k(x_1, \ldots, x_d)$ module in the obvious way. Then the image of $\chi_d$ in $k[x_1, \ldots, x_d, t]$ is $p(t) = n!((t - x_1)\cdots(t - x_d)$, where now the order of the terms does not matter. Hence $T$ satisfies $(T - x_1)\cdots(T - x_d) = 0$ in $M_n(S)$. So we can view $S^n$ as an $R = k[x_1, \ldots, x_d, t]/(p(t))$-module $M$. For each $i$ consider the quotient $S_i := R/(t - x_i)$, which is isomorphic to $S$ under the natural map $S \to S_i$. Define $M_i = M \otimes_R S_i$. Since the map $S \to S_1 \times \cdots \times S_d$
$S_d$ is an isomorphism after inverting $a = \prod_{i \neq j} (x_i - x_j)$ and $a$ is a nonzerodivisor on $M$, the natural map $M \to M_1 \oplus \cdots \oplus M_d$ is injective. Hence there is some $i$ such that $M_i$ has positive rank. Consider $M := M/(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_d)M$ and $M_i := M_i/(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)M_i$. Now since $M_i$ (is finitely generated and) has positive rank $M_i \neq 0$. Observe that since $M = S^d$, the natural map $k^d \to M$ is an isomorphism. Moreover the action of $t$ on $M$ is identified with the action of $X_t$. Now, $M_t = M/(t-x_1)M = M/(X_1-1)M \neq 0$. Hence $X_1-1$ is not invertible, $X_1-1$ has nonzero kernel, and $X_1$ has a 1-eigenvector.

(2.) Let us compute $\chi(x_1, \ldots, x_d, T)$. We denote by $\delta_j^i$ the Kronecker function. We have

\[
\chi_d(x_1, \ldots, x_d, T) = \sum_{\sigma \in S_d} \left( \prod_{i=1}^{d} x_i X_i - x_{\sigma(1)} \right) \cdots \left( \sum_{i=1}^{d} x_i X_i - x_{\sigma(d)} \right)
\]

\[
= \sum_{\sigma \in S_d} \prod_{j=1}^{d} \left( \sum_{i=1}^{d} x_i (X_i - \delta_{\sigma(j)}^i) \right) = \sum_{1 \leq i_1, \ldots, i_d \leq d} x_{i_1} \cdots x_{i_d} \left( \sum_{\sigma \in S_d} (X_i - \delta_{\sigma(1)}^i) \cdots (X_{i_d} - \delta_{\sigma(d)}^i) \right).
\]

In the second line the term order matters so the product is taken in the natural order $1, 2, \ldots, d$. Now suppose that $\chi_d(x_1, \ldots, x_d, T) = 0$. Then for all $1 \leq i_1, \ldots, i_d \leq d$ we have

\[
\sum_{\sigma \in S_d} (X_{i_1} - \delta_{\sigma(1)}^i) \cdots (X_{i_d} - \delta_{\sigma(d)}^i) = 0. \tag{2}
\]

For each $j \neq i$, we consider the noncommutative monomial $x_i x_j x_i^{d-2}$ and its equation (2),

\[
\sum_{\sigma \in S_d} (X_i - \delta_{\sigma(1)}^i)(X_j - \delta_{\sigma(2)}^j)(X_1 - \delta_{\sigma(3)}^i) \cdots (X_{i_d} - \delta_{\sigma(d)}^i) = 0. \tag{3}
\]

Note that since $X_i(m) = m$, we calculate

\[
(X_i - \delta_{\sigma(3)}^i) \cdots (X_i - \delta_{\sigma(d)}^i)m = \begin{cases} m & i \notin \{\sigma(3), \ldots, \sigma(d)\}, \\
0 & i \in \{\sigma(3), \ldots, \sigma(d)\}.
\end{cases}
\]

Therefore applying (3) to $m$ and simplifying we get

\[
\sum_{\sigma \in S_d, \sigma(1) = i} (X_i - 1)(X_j - \delta_{\sigma(2)}^i)m + \sum_{\sigma \in S_d, \sigma(2) = i} X_i X_j m = (d-1)!((X_i - 1)(X_j - \delta_{\sigma(2)}^i)m + X_i X_j m) = (d-1)!((X_i - 1)X_j m + X_i X_j m)
\]

\[
= (d-1)!((X_i - 1)X_j m)
\]

\[
= (d-1)!2X_i-1 X_j m
\]

\[
= 0,
\]

where passing from the first line to the second we use the fact that $(X_i - 1)\delta_{\sigma(2)}^i m = 0$.

Now, consider the special case of (2) corresponding to $x_i^d$:

\[
\sum_{\sigma \in S_d} (X_i - \delta_{\sigma(1)}^i) \cdots (X_i - \delta_{\sigma(d)}^i) = \sum_{j=1}^{d} \sum_{\sigma \in S_d, \sigma(j) = i} X_i^{j-1}(X_i - 1)X_i^{d-j-1} = d! X_i^{d-1}(X_i - 1) = 0.
\]

Since $X_i^{d-1}(X_i - 1) = 0$ it follows that $2X_i - 1$ is invertible. However, $(2X_i - 1)X_j m = 0$ so $X_j m = 0$.
Moreover, suppose \( n \in \mathbb{Q} \) to factor through subspace of \( k \). Let \( M_n(k, m) \subset M_n(k) \) be the algebra of operators that preserve \( k \). So if \( V \) then a codimension one subspace \( X \) algebraic multiplicity. So there is a 1-eigenvector \( k \) for \( k \) of \( (X \otimes k)_d \). Since \( \mathfrak{m} \) vanishes in \( \text{End} k \langle x_1, \ldots, x_d \rangle \). Now given \( \mathfrak{m} \subset k \) such that \( \mathfrak{m} \) is stable under the action of \( X_1, \ldots, X_d \). Let \( M_n(k, m) \subset M_n(k) \) be the algebra of operators that preserve \( k \). Then there is a surjective algebra homomorphism \( M_n(k, m) \to M_{n-1}(k) \). Since \( X_1, \ldots, X_d \in M_n(k, m) \) we find that \( T \in M_n(k, m) \otimes_k k \langle x_1, \ldots, x_d \rangle \). So if \( X_1', \ldots, X_d' \in M_{n-1}(k) \) are the images of \( X_1, \ldots, X_d \) then \( T' = x_1X_1' + \ldots + x_dX_d' \) satisfies \( \chi_d \).

By induction we see that \( (X_i')^2 = X_i' \) and \( X_i'X_j' = 0 \) for \( i \neq j \). In particular, there is a codimension 1 subspace preserved by \( X_1', \ldots, X_d' \). Its inverse image in \( k^n \) (we identify \( k^{n-1} \) with \( k^n/k \mathfrak{m} \)) is then a codimension one subspace \( V' \subset k^d \) which is invariant under \( X_1, \ldots, X_d \). Again by induction, \( X_i^2 - X_i \) and \( X_iX_j(i \neq j) \) annihilate \( V' \). There is some \( i \) such that \( X_i \) acts by the identity on \( k^n/V' \). Since \( X_i^d - 1 = 0 \), the geometric multiplicity of 1 as an eigenvalue of \( X_i \) is equal to its algebraic multiplicity. So there is a 1-eigenvector \( m \in k^n \) whose image in \( k^n/V' \) is nonzero. Again Lemma 5 implies that \( X_1m = 0 \) for \( j \neq i \). Hence the relations \( X_i^2 - X_i \) and \( X_iX_j \) annihilate a basis for \( k^n \) and hence annihilate \( k^n \).

\[ \Rightarrow \] Suppose that \( \phi \) is an algebra map. Then we have \( X_i^2 = X_i \) for all \( i \) and \( X_1X_i = 0 \) if \( i \neq j \). Decompose \( k^n = V_1 \oplus \ldots \oplus V_d \) where \( V_i = X_i(k^n) \). Then \( T \) preserves \( V_i \otimes k \langle x_1, \ldots, x_d \rangle \) for each \( i \). So we can view \( T \) as an element of \( \prod_{i=1}^d \text{End}_k(V_i) \otimes k \langle x_1, \ldots, x_d \rangle \). Since \( (T - x_i) \) vanishes identically on \( V_i \otimes k \langle x_1, \ldots, x_d \rangle \) we see that for each \( \sigma \in S_3 \) and each \( i \) the image of \( (T - x_{\sigma(1)}) \cdots (T - x_{\sigma(d)}) \) vanishes in \( \text{End}_k(V_i) \otimes k \langle x_1, \ldots, x_d \rangle \) and hence in \( M_n(k \langle x_1, \ldots, x_d \rangle) \). Since all of the terms of \( \chi_d(T) \) vanish in \( M_n(k \langle x_1, \ldots, x_d \rangle) \), so does \( \chi_d(T) \).

Questions

There are many natural questions that surround the notion of characteristic morphism. We point out a few of them.

**Question 1.** What are the irreducible characteristic morphisms for \( A = k^x \)? Are there infinitely many for \( d \geq 3 \)?

Replacing a commutative semisimple algebra with a semisimple algebra, Theorem A fails to hold. Indeed, the map \( \phi : \text{Mat}_d(k) \to \text{Mat}_r(k) \) defined by \( \phi(M) = MT \) is not a homomorphism, but does satisfy the hypotheses of Theorem A. Moreover, \( \phi \) is a characteristic morphism.

**Question 2.** Is there a characterization of when a linear map \( \phi : \text{Mat}_d(k) \to \text{Mat}_r(k) \) is a homomorphism along the lines of Theorem A?

Let \( V \) is a finite dimensional vector space and \( F(t) \in \text{Sym}^d(V^\vee) \) be monic and homogeneous. Given \( v \in V \) we can consider the image \( F_v(t) \) of \( F(t) \) under the homomorphism \( \text{Sym}^d(V^\vee) \to k[t] \) induced by \( v : V^\vee \to k \). The main theorem of [CK15] implies that there always exists a linear map \( \phi : V \to \text{Mat}_r(k) \) for some \( r \) such that \( F_v(\phi(v)) = 0 \) for all \( v \in V \). There is a natural non-commutative generalization of this problem.

**Question 3.** For which monic, homogeneous elements \( F(t) \) of \( \text{T}(V^\vee) \otimes k[t] \), does there exist an element \( \phi^\vee \in V^\vee \otimes \text{Mat}_r(V) \) for some \( r \) such that \( F(\phi^\vee) = 0 \) in \( \text{T}(V^\vee) \otimes \text{Mat}_r(k) \)?

If \( F(t) \) is the symmetrization of the characteristic polynomial of an algebra structure on \( V \) then we have an affirmative answer. However, if \( F(t) = t^2 - u \otimes v \) where \( u, v \) are linearly independent, then there is no such element.
References


