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Modularity of Galois representations and motives with good reduction properties

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Abstract. This article consists of rather informal musings about relationships between Galois representations, motives and automorphic forms. These are occasioned by recent progress on Serre's conjecture in the articles [50,44,51,52].

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1. Introduction

We would like to discuss the recent progress on Serre's conjecture and directions which seem interesting to pursue that emerge out of the proof. The article originates in some lectures the author gave in Boston and Paris between April–June 2006, and we retain the informal tone of these lectures. Even when we pretend to prove a result in this note, the proof is more illustrative than exhaustive.

The reader is referred to [88] and [47] for other expository articles about the work on Serre's conjecture.

Let $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ be a continuous, absolutely irreducible, two-dimensional, odd ($\det\bar{\rho}(c) = -1$ for c a complex conjugation) representation, with $\mathbb{F}$ a finite field of characteristic p. We hold the prime p fixed in our discussion. We say that such a representation is of *Serre-type*, or S-type, for short.

Serre has conjectured in [66] that every $\bar{\rho}$ of S-type is *modular*, i.e., *arises from* (with respect to some fixed embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$) a newform f of some weight $k \geq 2$ and level N prime to p. We fix embeddings $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ for all primes p hereafter, and when we say (a place above) p, we will mean the

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place induced by this embedding. By *arises from* f we mean that there is an integral model $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$ of the p-adic representation ρ_f associated to f, such that $\bar{\rho}$ is isomorphic to the reduction of ρ modulo the maximal ideal of $\mathcal{O}$.

$$\begin{array}{ccc} G_{\mathbb{Q}} & \xrightarrow{\rho_f} & \mathrm{GL}_2(\mathcal{O}) \\ & \overset{\bar{\rho}}{\searrow} & \downarrow \\ & & \mathrm{GL}_2(\mathbb{F}) \end{array}$$

The conjecture in [66], that we shall call the S-conjecture, is more precise. We can split it in two parts:

1. *Qualitative form:* In this form, one only asks that $\bar{\rho}$ of S-type arise from a newform of unspecified level and weight.
2. *Refined form:* In this form, one asks that $\bar{\rho}$ of S-type arise from a newform $f \in S_{k(\bar{\rho})}(\Gamma_1(N(\bar{\rho})))$ where $N(\bar{\rho})$ and $k(\bar{\rho})$ are defined as below.

The integer $N(\bar{\rho})$ is the (prime to p) Artin conductor of $\bar{\rho}$, and $k(\bar{\rho})$ the weight of $\bar{\rho}$ as defined in [66, Section 1.2, Section 2]. The invariant $N(\bar{\rho})$ is defined using $(\bar{\rho}|_{I_\ell})_{\ell \neq p}$, and is divisible exactly by the primes ramified in $\bar{\rho}$ that are other than p. The weight $k(\bar{\rho})$ is such that $2 \leq k(\bar{\rho}) \leq p^2 - 1$ if $p \neq 2$ ($2 \leq k(\bar{\rho}) \leq 4$ if $p = 2$), and is defined using $\bar{\rho}|_{I_p}$. To give a flavor of the definition of $k(\bar{\rho})$ in [66], if $p > 2$ and $\bar{\chi}_p$ is the mod p cyclotomic character, and $\bar{\rho}|_{I_p}$ is of the form

$$\begin{pmatrix} \bar{\chi}_p^a & * \\ 0 & 1 \end{pmatrix},$$

for $1 \leq a \leq p - 1$, then $k(\bar{\rho}) = a + 1$, unless $a = 1$. In the latter case $k(\bar{\rho}) = 2$ if $\bar{\rho}|_{I_p}$ arises from a finite flat group scheme over $\mathbb{Z}_p^{\mathrm{nr}}$ (the ring of integers of the maximal unramified extension of $\mathbb{Q}_p$), and is otherwise $p + 1$. It is an important feature of the weight $k(\bar{\rho})$, for $p > 2$, that if $\overline{\chi_p}$ is the mod p cyclotomic character, then for some $i \in \mathbb{Z}$, we have $2 \leq k(\bar{\rho} \otimes \overline{\chi_p}^i) \leq p + 1$. This allows one to assume that $k(\bar{\rho}) \leq p + 1$ when attempting to prove the S-conjecture. The theory of Fontaine-Laffaille [31] may then be used to study such $\bar{\rho}$.

An extensive body of work of several mathematicians, Ribet, Mazur, Carayol, Gross, Coleman-Voloch, Edixhoven, Diamond et al., see [58], proves that for $p > 2$ the qualitative form implies the refined form.

Serre in fact made an even more precise conjecture. He defined a character $\varepsilon(\bar{\rho})$, the Teichmüller lift of $\overline{\varepsilon}(\bar{\rho}) := \det(\bar{\rho})\chi_p^{1-k(\bar{\rho})}$, and asserted that $\bar{\rho}$ arises from a newform in $S_{k(\bar{\rho})}(\Gamma_0(N(\bar{\rho})), \varepsilon(\bar{\rho}))$. This turned out to be inaccurate for primes $p = 2, 3$, when the representation considered is induced from a character of $G_{\mathbb{Q}(i)}$ and $G_{\mathbb{Q}(\sqrt{-3})}$ respectively. In all other cases it is proved in

[12] that if $\bar{\rho}$ arises from $S_{k(\bar{\rho})}(\Gamma_1(N(\bar{\rho}))$, then it also arises from a newform in $S_{k(\bar{\rho})}(\Gamma_0(N(\bar{\rho})), \varepsilon(\bar{\rho}))$.

2. Historical backdrop

Serre made his conjecture when Deligne had attached Galois representations to higher weight (≥ 2) newforms (generalising the work of Eichler-Shimura and proving a conjecture of Serre in [67]) and Swinnerton-Dyer [76] and Serre [68] had been studying the properties of these representations.

For instance, consider the Ramanujan Δ function, which is up to scalars the unique non-zero cusp form of weight 12 on $\mathrm{SL}_2(\mathbb{Z})$:

$$\Delta(z) = q\Pi(1-q^n)^{24} = \Sigma\tau(n)q^n$$

with $q = e^{2\pi i z}$. This is an eigenform for the Hecke operators T_ℓ for each prime ℓ, where the action of T_ℓ is given by:

$$\Delta(z)|T_\ell = \Sigma\tau(n\ell)q^n + \ell^{11}\Sigma\tau(n)q^{n\ell},$$

and the fact that Δ is an eigenfunction means that

$$\Delta(z)|T_\ell = \tau(\ell)\Delta(z).$$

For each prime p there is an attached Galois representation

$$\rho_{\Delta,p} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}_p)$$

which is irreducible for all p, and which is unramified outside p. This is characterised for $p \neq 2, 3, 5, 7, 691$, by the main theorem of [13], by the property that the characteristic polynomial of $\rho_{\Delta,p}(\mathrm{Frob}_\ell)$ for all primes $\ell \neq p$ is $X^2 - \tau(\ell)X + \ell^{11}$.

Swinnerton-Dyer and Serre proved in [76,69] that this representation has large image, i.e., $\rho_{\Delta,p}(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}))$ contains $\mathrm{SL}_2(\mathbb{Z}_p)$ for all p different from

$$2, 3, 5, 7, 23, 691.$$

This they did by first proving that the mod p image contains $\mathrm{SL}_2(\mathbb{F}_p)$ for these primes. This suffices because of [72, Section 3.4, Lemma 3]:

Lemma 2.1. *For $p > 3$ a closed subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ that contains $\mathrm{SL}_2(\mathbb{F}_p)$ in its reduction mod p, contains $\mathrm{SL}_2(\mathbb{Z}_p)$.*

This lemma has been greatly generalised by Vasiu (see [83, Theorem 3.5]).

For all primes the image is an open subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ (and in fact the image of $\Pi_p\rho_{\Delta,p}$ contains an open subgroup of $\Pi_p\mathrm{SL}_2(\mathbb{Z}_p)$).

In proving these results, the main tool they used was the study of congruences between modular forms which has been an intense focus of research ever since.

For instance consider the following formal manipulation due to Swinnerton-Dyer, see [67, Section 3.2]:

$$\Delta(z) = q\Pi(1-q^n)^{24} = q\Pi(1-q^n)^2\Pi(1-q^n)^{22}$$
$$\equiv q\Pi(1-q^n)^2\Pi(1-q^{11n})^2 \pmod{11}$$

and the latter is the q-expansion of the unique cusp form in $S_2(\Gamma_0(11))$.

Serre proved in [68] that there is a Hecke invariant isomorphism between $S_{p+1}(\mathrm{SL}_2(\mathbb{Z}), \mathbb{F})$ and $S_2(\Gamma_0(p), \mathbb{F})$ where $\mathbb{F}$ is a finite field of characteristic p. As for $p = 11$, both sides are one-dimensional and spanned by the reduction of Δ and the cusp form attached to the unique (up to isogeny) elliptic curve E over $\mathbb{Q}$ of conductor 11 respectively, we see that the congruence noted above is a particular case of Serre's result. This result of Serre guided some of the work in [50] where it is shown that a $\bar{\rho}$ of S-type with $k(\bar{\rho}) = p+1$ can be lifted to a 2-dimensional, *geometric* p-adic representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$ that is crystalline of weight $p+1$ at p, and also to one that is semistable of weight 2 at p.

We give a brief explanation of the terms used in the previous sentence that will also be used freely in what follows. By a *geometric* representation we mean following [32] that the p-adic representation is ramified at a finite set of primes and is potentially semistable at p (a property of $\rho|_{D_p}$) in the sense of Fontaine's theory, [30], that also defines the property of the representation $\rho|_{D_p}$ being crystalline and semistable. Such representations are Hodge-Tate and thus have Hodge-Tate weights attached to them (see [30]). We say that a Hodge-Tate representation $G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$ is of weight k for some non-negative integer k if its Hodge-Tate weights are $(k-1, 0)$.

It is in this context of the study of Galois representations attached to newforms, and congruences, that Serre asked for the converse direction.

3. Main theorem

The following theorem is proved in [51] and [52] building on the work of [50] and [44].

Theorem 3.1.

(i) *For* $p > 2$ *the S-conjecture is true for odd conductors, i.e., for* $\bar{\rho}$ *unramified at 2.*

(ii) *For* $p = 2$ *the S-conjecture is true when* $k(\bar{\rho}) = 2$.

A corollary of Theorem 3.1 is that the qualitative form of the S-conjecture implies its refined form as it fills in a missing case in characteristic 2 not covered in [11] and [85]. The missing case was that of $\bar{\rho}$ of S-type in characteristic 2, the projective image of $\bar{\rho}$ is not dihedral and $\bar{\rho}|_{D_2}$ takes values in scalar matrices and thus $k(\bar{\rho}) = 2$.

The restrictions in the theorem are for technical reasons, and it is expected that the methods used in the proof of Theorem 3.1 will eventually suffice to prove all of the S-conjecture. In this direction it is proved in [51] that the general case of the S-conjecture follows from the following:

Hypothesis (H). Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$ be a continuous, odd, irreducible, p-adic representation, such that:

(i) the residual representation $\bar{\rho}$ has non-solvable image, and $\bar{\rho}$ is modular;
(ii) ρ is unramified outside a finite set of primes, is potentially crystalline at p and of weight 2.

Then ρ is modular.

Kisin's results in [40] prove (H) for $p > 2$ and are an important ingredient in the proof of Theorem 3.1. Kisin is expected to prove (H) for $p = 2$ in the near future generalising the methods he had introduced in [40]. The case of (H) when $p = 2$ and ρ is crystalline of weight 2 at 2 is done in [52], and is again an important ingredient in the proof of Theorem 3.1. When stating consequences of the S-conjecture we will assume that it has been proved fully.

The main steps of the proof of Theorem 3.1 are:

(1) Existence of lifts of $\bar{\rho}$ to p-adic representations ρ with prescribed ramification properties (see Section 5 below). This is done by a technique introduced in [50] and developed in [44] and [52].
(2) Making the lifts ρ part of compatible systems: this is due to Taylor and Dieulefait (see [79] and [21], and also [89]).
(3) Use of results of Tate, Fontaine, Brumer-Kramer and Schoof (see [77,28,10,64]) to prove the S-conjecture for small values of $k(\bar{\rho}), N(\bar{\rho})$.

The utility of compatible systems in conjunction with results of Tate (or Fontaine) was noticed by Dieulefait (see [21]): it was also noticed later by Wintenberger independently (see [87]).

The key technical ingredients of the proof are:

(i) Modularity lifting ($R = \mathbb{T}$) theorems due to Wiles, Taylor-Wiles, Diamond, Fujiwara, Skinner-Wiles, Savitt, Kisin (see [86,81,17,26,73–75,62,40]). The developments in [40] of the method of [86,81] are of critical importance in [52].
(ii) The potential version of the S-conjecture proved by Taylor in [79,80].

(iii) Presentation results for certain deformation rings due to Mazur, Böckle and Kisin (see [54,3,4] and [42]). These results were first proved by Mazur in his pioneering work where he introduced the formalism of deformation theory for studying Galois representations. Mazur proved presentation results for unrestricted global deformation rings using obstruction theory arguments, Böckle refined these to prove such results for global deformation rings with local conditions. Kisin relativised these arguments by considering the global deformation rings as algebras over local deformation rings.

4. Lifting modular forms

There are two different ways of formulating the S-conjecture. The way it was formulated in [66] was in terms of representations arising as reductions of the p-adic representations associated to newforms. One can also work in the context of Katz mod p modular forms as was suggested by Serre in [66], and taken up by Edixhoven (see [22,23]).

The contents of this section are meant to provide one possible motivation for the next section.

Let N and k be positive integers and let R be a $\mathbb{Z}[\frac{1}{N}]$-algebra. Katz has defined in [37] the space of modular forms of weight k and level N over R, that we denote by $S_k(\Gamma_1(N), R)$ (see [23] for an exposition), as functions with certain properties on "test objects".

While we will not recall the general definition, let us recall it in a special, representative case in which the definition is more down-to-earth.

Let $N > 4$ and let $\pi : \mathcal{E}_{/R} \to X_1(N)_{/R}$ with $\mathcal{E}_{/R}$ the universal generalised elliptic curve, and 0 be the zero section. We have the invertible $\mathcal{O}_{X_1(N)_{/R}}$-module $\omega_R = 0^*(\Omega_{\mathcal{E}_{/R}/X_1(N)_{/R}})$ obtained by pulling back the sheaf of Kähler differentials of $\mathcal{E}_{/R}$ over $X_1(N)_{/R}$ by the zero-section. (We will often drop R from the notation if it is understood what R is.) For a positive integer k we denote by $\omega^{\otimes k}$ by ω^k. Let D be the reduced divisor of cusps on $X_1(N)$. Then define $S_k(\Gamma_1(N), R) := H^0(X_1(N)_{/R}, \omega^k(-D))$. The space $S_k(\Gamma_1(N), R)$ has an action of the Hecke operators T_n for all $n \in \mathbb{N}$ (sometimes for $\ell | N$ these T_ℓ's are denoted by U_ℓ) and the diamond operator $\langle a \rangle$ for $a \in (\mathbb{Z}/N\mathbb{Z})^*$. For a character $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^* \to R^*$ we denote by $S_k(\Gamma_0(N), \varepsilon; R)$ the R-submodule on which $(\mathbb{Z}/N\mathbb{Z})^*$ acts via ε.

Let p be a prime $(p, N) = 1$, and let $\mathbb{F}$ be a finite field of characteristic p. If $f \in S_k(\Gamma_1(N), \mathbb{F})$ ($k \geq 2$) is an eigenform for these operators the construction of Deligne associates a Galois representation $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ in the mod p étale cohomology of $X_1(N)_{/\mathbb{Q}}$ with the standard properties. For $k = 1$ we also have such a representation which may be deduced (by multiplication by the Hasse invariant) from the representations associated to weight p forms.

Edixhoven modified Serre's definition of weights slightly to define another invariant $k_{\bar\rho}$ which was the same as $k(\bar\rho)$ except when:

(i) $\bar\rho$ is unramified at p, in which case $k_{\bar\rho} = 1$ and $k(\bar\rho) = p$
(ii) $p = 2$ and $\bar\rho$ is not finite at 2, in which case $k_{\bar\rho} = 3$ and $k(\bar\rho) = 4$.

Thus the conjecture as modified in Edixhoven's articles [22,23] is:

Characteristic p form of the S-conjecture. If $\bar\rho$ is of S-type, it arises from $S_{k_{\bar\rho}}(\Gamma_0(N(\bar\rho)), \bar\varepsilon(\bar\rho), \mathbb{F})$.

To relate the 2 conjectures we have the following propositions:

Proposition 4.1.

1. *Suppose that* $N > 4, k \geq 1$, $(N, p) = 1$. *Then the map* $S_k(\Gamma_1(N), \overline{\mathbb{Z}}_p) \to S_k(\Gamma_1(N), \overline{\mathbb{F}}_p)$ *is surjective for almost all primes p.*
2. *Suppose that* $N > 4, k \geq 2$, $(N, p) = 1$. *Then the map* $S_k(\Gamma_1(N), \overline{\mathbb{Z}}_p) \to S_k(\Gamma_1(N), \overline{\mathbb{F}}_p)$ *is surjective.*

For part (1) it is enough to prove the statement $S_k(\Gamma_1(N), \mathbb{Z}_p) \to S_k(\Gamma_1(N), \mathbb{F}_p)$ is surjective for almost all primes p. The hypothesis $N > 4$ ensures that we have an exact sequence of sheaves

$$0 \to \omega^k_{\mathbb{Z}_p}(-D) \to \omega^k_{\mathbb{Z}_p}(-D) \to \omega^k_{\mathbb{F}_p}(-D) \to 0,$$

where the second map is multiplication by p ($(p, N) = 1$). Thus looking at the corresponding long exact sequence of cohomology groups we see that the map $S_k(\Gamma_1(N), \mathbb{Z}_p) \to S_k(\Gamma_1(N), \mathbb{F}_p)$ is surjective if

$$H^1(X_1(N)_{/\mathbb{Z}_p}, \omega^k(-D))[p] = 0.$$

Note that

$$H^1(X_1(N)_{/\mathbb{Z}_p}, \omega^k(-D)) = H^1(X_1(N)_{/\mathbb{Z}[\frac{1}{N}]}, \omega^k(-D)) \otimes_{\mathbb{Z}[\frac{1}{N}]} \mathbb{Z}_p.$$

As $H^1(X_1(N)_{/\mathbb{Z}[\frac{1}{N}]}, \omega^k(-D))$ is a finitely generated $\mathbb{Z}[\frac{1}{N}]$-module, it is p-torsion free for almost all primes p which proves the first part.

For the second part we see by Nakayama's lemma and the associated long exact sequence that it will be enough to prove

$$H^1(X_1(N)_{/\mathbb{F}_p}, \omega^k(-D)) = 0.$$

By duality this is isomorphic to $H^0(X_1(N)_{/\mathbb{F}_p}, \omega^{2-k})$ which vanishes if $k > 2$. The case $k = 2$ follows by considering lifts to differentials with one simple pole at say a point P (see [13]) or by using that the Kodaira-Spencer map identifies weight 2 forms with differentials and these can be lifted (see [23]).

Remark. As just seen, in the weight one case the obstruction to liftings occurs for primes p in the support of $H^1(X_1(N)_{\mathbb{Z}[\frac{1}{N}]}, \omega(-D))$. This seems like a sporadic (for example difficult to identify in any *a priori* way) set of primes. It would be of interest to bound such sporadic primes in terms of the arithmetic of $X_1(N)$(for example its Faltings height).

Proposition 4.2. *Suppose that $N > 4, k \geq 2$, $(N,p) = 1$ and ε a character of $(\mathbb{Z}/N\mathbb{Z})^* \rightarrow \overline{\mathbb{Z}}_p^{\,*}$, with reduction $\overline{\varepsilon}$, such that $\varepsilon(-1) = (-1)^k$. Let m be a maximal ideal of the Hecke algebra with $p \in \mathsf{m}$ acting on $S_k(\Gamma_0(N), \overline{\varepsilon}, \overline{\mathbb{F}}_p)$. Assume that m is non-Eisenstein, i.e., the corresponding semisimple Galois representation $\bar{\rho}_{\mathsf{m}}$ is irreducible. Further assume $\bar{\rho}_{\mathsf{m}}$ is not induced from $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$ when $p = 2, 3$ respectively. Then m is the image of a maximal ideal, that we again denote by m, of the Hecke algebra acting on $S_k(\Gamma_0(N), \varepsilon, \overline{\mathbb{Z}}_p)$ and the reduction map $S_k(\Gamma_0(N), \varepsilon, \overline{\mathbb{Z}}_p)_{\mathsf{m}} \rightarrow S_k(\Gamma_0(N), \overline{\varepsilon}, \overline{\mathbb{F}}_p)_{\mathsf{m}}$ is surjective.*

We sketch the proof (see [23, Proposition 1.10] and page 4 of [24]). This is a result of Carayol (see [12]) who proved it in the context of étale cohomology groups, and Serre (in a course in Collège de France, 1987–88) who proved it when considering certain coherent cohomology groups. Our viewpoint here is the latter while the proof we use is closer to [12]. Fix the ring of integers of a large extension of $\mathbb{Q}_p$, say $\mathcal{O}$, with residue field $\mathbb{F}$. Consider the twisted sheaves $\omega^k_{\mathcal{O}}(\varepsilon)$ and $\omega^k_{\mathbb{F}}(\overline{\varepsilon})$ which arise from the morphism $r : X_1(N) \rightarrow X_0(N)$. [Thus $\omega^k_{\mathcal{O}}(\varepsilon)$ is the $\mathcal{O}$-submodule of $r_*(\omega^k_{\mathcal{O}})$ on which $(\mathbb{Z}/N\mathbb{Z})^*$ acts *via* ε.] The cokernel of the reduction map $\omega^k_{\mathcal{O}}(\varepsilon) \rightarrow \omega^k_{\mathbb{F}}(\overline{\varepsilon})$ is a skyscraper sheaf on $X_0(N)$ supported on points whose automorphism group is of order 4 or 6 depending on whether the characteristic of $\mathbb{F}$ is 2 or 3, and on the cusps. Then Serre observes that for primes ℓ congruent to -1 modulo 4 (resp. modulo 3) there is no elliptic curve with an automorphism of order 4 (resp. 6) fixing a subgroup of order ℓ. From this we may conclude that the non-Eisenstein maximal ideals that occur in the support of the Hecke action on the (global sections of) skyscraper sheaf above are such that for such primes ℓ, $T_\ell \in \mathsf{m}$. Let m be a maximal ideal of the Hecke algebra acting on $H^0(X_0(N)_{/\mathbb{F}}, \omega(\overline{\varepsilon}))_{\mathsf{m}}$ such that $\bar{\rho}_{\mathsf{m}}$ is not induced from $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$ when $p = 2, 3$ respectively. Then we deduce that the map $H^0(X_0(N)_{/\mathcal{O}}, \omega(\varepsilon)) \rightarrow H^0(X_0(N)_{/\mathbb{F}}, \omega(\overline{\varepsilon}))_{\mathsf{m}}$ (given by composition of the reduction map with the projection to the m-primary component) is surjective. If D denotes the reduced divisor of cusps of $X_0(N)$, note that the Hecke action on the cokernel of $\omega^k_{\mathcal{O}}(\varepsilon)(-D) \rightarrow \omega^k_{\mathcal{O}}(\varepsilon)$ is Eisenstein. Further, $S_k(\Gamma_0(N), \varepsilon, \mathcal{O})$, resp. $S_k(\Gamma_0(N), \overline{\varepsilon}, \mathbb{F})$, are global sections of $\omega^k_{\mathcal{O}}(\varepsilon)(-D)$ and $\omega^k_{\mathbb{F}}(\overline{\varepsilon})(-D)$ respectively. Altogether we have proven the proposition. (Note that if the character ε kills the 4 or 6 torsion of $(\mathbb{Z}/N\mathbb{Z})^*$ we may conclude after localising only at a non-Eisenstein m. This type of observation is sometimes useful: see [52, Sections 6.2,6.3] for such a use in a different context.)

The propositions make plausible the statement that for $p > 3$ the characteristic p form of the S-conjecture implies the original form. Proposition 4.1 allows one to show that if $\bar{\rho}$ arises from $S_k(\bar{\rho})(\Gamma_1(N(\bar{\rho})), \mathbb{F})$, then it also arises from the space of characteristic 0 forms $S_k(\bar{\rho})(\Gamma_1(N(\bar{\rho})))$. Proposition 4.2 refines this to yield that if $\bar{\rho}$ arises from $S_k(\bar{\rho})(\Gamma_0(N(\bar{\rho})), \bar{\varepsilon}(\bar{\rho}), \mathbb{F})$, then it also arises from the space of characteristic 0 forms $S_k(\bar{\rho})(\Gamma_0(N(\bar{\rho})), \varepsilon(\bar{\rho}))$. One gets rid of the assumption of $N > 4$ by increasing levels and looking at invariants: see [23] for details. For $p = 2, 3$ we have to make exceptions of representations that are induced from $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-3})$ repectively.

The characteristic p form is now known for $p > 2$, and for $p = 2$ it is known (see [23] and the remark after Theorem 3.1) in all cases except in one exceptional case when the semisimplification of $\bar{\rho}|_{D_2}$ is the direct sum of two equal characters.

In conclusion we may say that the difference between Serre's original formulation of his conjecture and the characteristic p form above is not so serious for weights > 1, but it is serious for weight 1. We meet similar phenomena in the next section.

4.1 Non-liftable weight one forms

There are examples of mod p weight one forms that cannot be lifted to characteristic zero weight one forms. We preface these examples by recalling that holomorphic weight one newforms give rise to odd, irreducible, 2-dimensional complex (Artin) representations of $G_{\mathbb{Q}}$.

(i) Mestre in a letter to Serre in October 1987 gave an example of a weight 1 mod 2 form whose attached Galois representation has image $\mathrm{SL}_2(\mathbb{F}_8)$ (the letter is reproduced in Mestre's appendix to [24]): this cannot be lifted to a characteristic 0 weight one form as the finite subgroups of $\mathrm{GL}_2(\mathbb{C})$ are limited. (G. Wiese informs us that in level 82, K. Buzzard found a mod 199 modular form of weight one whose associated Galois representation has image which contains $\mathrm{SL}_2(\mathbb{F}_{199^2})$.)

(ii) Consider a 2-dimensional mod 2 representation $\bar{\rho}$ of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ that is unramified at 2, has projective image $A_5 \simeq \mathrm{SL}_2(\mathbb{F}_4)$, and $\bar{\rho}(c)$ is the indentity. There are examples of such A_5-extensions of $\mathbb{Q}$ that are totally real and of discriminants 8311^2 and 13613^2 on page 219 of [66]. This according to the characteristic 2 version of Serre's conjecture arises from a weight 1 mod 2 form f which cannot be lifted, although the image of $\bar{\rho}$ can be lifted. This is because if there were a lifting to $\tilde{\rho}$ of ρ, with $\tilde{\rho}$ arising from a classical weight one form, then the reduction map from the projective image of $\tilde{\rho}$ to that of $\bar{\rho}$ would be an isomorphism which would contradict the fact that $\tilde{\rho}$ is odd. Thus the obstruction is a local one at infinity. Note that in the examples above G. Wiese informs us that the conjugacy class of the Frobenius at 2 in the A_5-extension is of odd order. Thus combining Theorem 3.1 (ii) and [35], one knows that the corresponding $\bar{\rho}$ arises from a characteristic 2 form of weight 1.

(iii) In the introduction of [85] some examples are given of weight one mod 2 representations, induced from a character of $G_{\mathbb{Q}(\sqrt{229})}$, that cannot be lifted to characteristic 0 weight 1 forms keeping the level fixed. On the other hand, by an *astuce* of Serre (Lemma 2 of [85]), these dihedral-type forms can always be lifted to characteristic 0 weight one dihedral-type holomorphic weight one forms of some (augmented) level.

5. Lifting Galois representations

5.1 Minimal liftings

Now we are motivated to look at lifting Galois representations.

The S-conjecture predicts liftings of Galois representations of S-type, i.e. a $\bar{\rho}$ of S-type (of weight $\leq p+1$ when $p > 2$) arises as the reduction of a p-adic representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$ such that $N(\rho) = N(\bar{\rho})$ and ρ is crystalline at p of Hodge-Tate weights $(k(\bar{\rho}) - 1, 0)$. This was proved when $\bar{\rho}$ has non-solvable image and $p > 2$ in [50]. In [51] and [52] this was proven also for $p = 2$ when the image of $\bar{\rho}$ is not dihedral and $k(\bar{\rho}) = 2$.

In fact *minimal liftings* (liftings with "minimal ramification") are constructed in [50] and [52] which are a little stronger than the above and which for example also have the property that $\det(\rho) = \varepsilon(\bar{\rho})\chi_p^{k(\bar{\rho})-1}$.

Note that when $\bar{\rho}$ is a mod 2 or mod 3 representation induced from $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$ one does not expect minimal liftings. The difficulties these representations pose in the contrasting optics of modular forms and Galois representations can be explained by the fact that there are no auxiliary primes in the sense of [18, Lemma 2] or [86] (see [19, Theorem 2.49]). Note that [18, Lemma 2] may be used to give another proof of Proposition 4.2 by imposing auxiliary level structure at a prime r such that there are no lifts of $\bar{\rho}$ that are ramified at r. Such primes r do not exist in the exceptional case. Neither do the auxiliary primes required by the method of [86] when $\bar{\rho}|_{\mathbb{Q}(\mu_{p^n})}$ is reducible.

When $k_{\bar{\rho}} = 1$ there is no serious hope in general of lifting $\bar{\rho}$ to a crystalline representation at p of Hodge-Tate weights $(0, 0)$ which by a result of Sen and Fontaine is equivalent to saying that the lift is unramified at p. The Fontaine-Mazur conjecture [32] predicts then that ρ has finite image, and as there are examples of Mestre of such $\bar{\rho}$ with large, non-solvable image (for example containing $\mathrm{SL}_2(\mathbb{F})$ with $|\mathbb{F}| > 5$), the classification of finite order subgroups of $\mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ will then forbid this.

While lifting methods might replace ([50] for the level aspect, and [33] for the weight aspect) geometric techniques used to show that the qualitative form of the S-conjecture implies its refined form (in its orginally stated from), to show that it implies the characteristic p form makes inevitable the use of techniques of arithmetic geometry as in [35] and [15] (when $k_{\bar{\rho}} = 1$).

The method of Gee produces companion forms using lifting techniques. But to deduce from this that a modular $\bar{\rho}$ of S-type, unramified at p with $\bar{\rho}(\mathrm{Frob}_p)$ having distinct eigenvalues, arises from a Katz weight one form of level N, one needs some geometric input. For example we may deduce this from:

1. The exact sequence of [38] (see also [24]):

$$0 \to S_1(\Gamma_1(N), \mathbb{F}_p) {\to} S_p(\Gamma_1(N), \mathbb{F}_p) {\to} S_{p+2}(\Gamma_1(N), \mathbb{F}_p).$$

   The first map is the Frobenius map F of [24], while the second is the Ramanujan operator Θ.
2. The existence of companion forms gives the existence of a form $f = \Sigma_n a_n q^n \in S_p(\Gamma_1(N), \mathbb{F}_p)$ such that $a_n = 0$ if $(n,p) = 1$ and f gives rise to $\bar{\rho}$ for an N prime to p.

In the case when $\bar{\rho}(\mathrm{Frob}_p)$ has multiple eigenvalues one has to use the methods of [15] valid for $p > 2$.

The extra information of when a $\bar{\rho}$ of S-types arises from a mod p weight one form is very valuable as it allows us to deduce Artin's conjecture from the S-conjecture as we recall below. But it is Serre's original definition of weights which is more in keeping with the techniques of the proof of the S-conjecture. Even in the other anomalous case when $p = 2, k_{\bar{\rho}} = 3, k(\bar{\rho}) = 4$, it is Serre's original definition which is better for the lifting techniques of the proof as the lifts constructed have to be odd.

5.2 Liftings with prescribed types

Consider a residual representation $\bar{\rho} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F})$ of S-type such that $\mathrm{im}(\bar{\rho})$ is non-solvable. Fix a large finite extension $E/\mathbb{Q}_p$, and its ring of integers $\mathcal{O}$.

Consider *lifting data* of the following kind:

For each prime $q \neq p$ a lifting $\tilde{\rho}_q$ of $\bar{\rho}|_{D_q}$ with values in $\mathrm{GL}_2(\mathcal{O})$, such that the lifts are unramified for almost all q. For $q = p$ fix a potentially semistable lift $\tilde{\rho}_p$ of $\bar{\rho}|_{D_p}$ with values in $\mathrm{GL}_2(\mathcal{O})$: attached to this there is a Weil-Deligne parameter (τ, N) with τ a finite order $\mathrm{GL}_2(\overline{\mathbb{Q}}_p)$-valued representation of I_p and N a nilpotent matrix in $\mathrm{GL}_2(\overline{\mathbb{Q}}_p)$, and with Hodge-Tate numbers (a, b) which we assume are unequal.

Also assume that there is an odd Hecke character ψ which matches with the determinants of $\tilde{\rho}_q|_{I_q}$ (this is only a condition for $p = 2$).

One of the main technical tools in the proof of Serre's conjecture in [50,44,51,52] is the following kind of result:

Congruences of Galois representations. After enlarging E, there is a lift $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O})$ of $\bar{\rho}$ with determinant arising from ψ, such that for all primes $q \neq p$, $\rho|_{I_q}$ is isomorphic to $\tilde{\rho}_q|_{I_q}$ (as E-valued representations and up to semisimplification), and $\rho|_{D_p}$ is potentially semistable such that its inertial Weil-Deligne parameter is isomorphic to $(\tau|_{I_p}, N')$ (for some nilpotent N') and with Hodge-Tate numbers (a, b).

Note that the lifts ρ are geometric in the sense of [32]. We do not quite prove this result: one needs this, and we have proved this, with many more restrictions at p. We prove enough of it to suffice for our needs. Such liftings with the properties above ensured for almost all primes q is a result of Ramakrishna [57]. For our applications it is important to produce the above more calibrated type of liftings with the properties ensured at *all* primes q.

When $\bar{\rho}$ is modular this is a result of Diamond and Taylor [18] building on work of Ribet [59], Carayol [12] (in the $\ell \neq p$ case) and the author [46] (in the $\ell = p$ case). In [46] the author used the weight part of Serre's conjecture to analyse the types at p of newforms which give rise to $\bar{\rho}$. In [33] the procedure is reversed.

The lifting method of [50] uses the following ingredients:

(i) $R = \mathbb{T}$ theorems only in the minimal case (the first such result is that of [81]),
(ii) potential modularity results of the type proven in [79,80], and
(iii) Böckle's presentation results [3].

The reliance on $R = \mathbb{T}$ theorems in the minimal case is not explictly spelled out in [50]: as a hint the reader may look at [44, Lemma 4.2]. On the other hand this is made explicit in the course of the proof of [51, Theorem 5.1] given in [52].

Assuming $\bar{\rho}$ is modular one needs only the first and third ingredients, which are available now in some generality (see [14] for the first, [42,43,34] for the third). This fact is used in [52] to prove level-raising results when $p = 2$, when combined with the observation that, mod 2, in the block of the Steinberg representation of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ for ℓ an odd prime (in the sense of [84]), there is a supercuspidal representation.

As the lifting method of [50] is axiomatic, given some local calculations of deformation rings (see [43] and [34]) it generalises to other situations too. For example, in [34] this type of argument is used to deduce analogs of [18] in more general situations when combined with results of [82].

We make a philosophical remark. It does not seem reasonable to get a geometric lifting of $\bar{\rho}$ that interpolates all of $\tilde{\rho}_q$, rather than just $\tilde{\rho}_q|_{I_q}$, for even one prime q, or interpolates all of $\tilde{\rho}_p|_{I_p}$. For instance in the case $\tilde{\rho}_q$ is unramified at q, the eigenvalues of $\tilde{\rho}_q(\mathrm{Frob}_q)$ are data that is part of the "continuous spectrum" of the corresponding local deformation rings (while $\tilde{\rho}_q|_{I_q}$ or the WD parameter of $\tilde{\rho}_p|_{I_p}$ are discrete) which allow these local deformation rings to be large enough to satisfy the axiomatics of the lifting method of [50,44,51]. Thus it is difficult to get liftings that fix the unramified parameters. Furthermore by the conjectures in [32] one would have to specify algebraicity, and purity, for the eigenvalues of $\tilde{\rho}_q(\mathrm{Frob}_q)$. For a treatment of a related issue for modular forms see [48], where again techniques using trace formula methods cannot handle construction of automorphic forms with certain specified Satake parameters at a place.

5.3 Method of producing liftings

(see [50,44,51,52]) We consider a deformation ring R (a complete, Noetherian, local $\mathcal{O}$-algebra), defined in terms of the lifting data, which has the property such that whenever there is a morphism $\pi : R \to \mathcal{O}'$, with $\mathcal{O}'$ the ring of integers of a finite extension $E'/\mathbb{Q}_p$, then there is a lifting $\rho_\pi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}')$ with the desired local properties (and conversely). Thus it will be enough to prove the existence of such a morphism π. The definition of this ring R is after Mazur, and also needs Kisin's idea of framed deformations.

Using Taylor's results on potential modularity of $\bar{\rho}$, and modularity lifting theorems ($R = \mathbb{T}$ theorems) over totally real fields, and a simple algebraic argument ([50, Lemma 2.4]) one first proves that R is a finite $\mathcal{O}$-module. Using obstruction-theory arguments of the type used by Mazur, Böckle and Kisin, one shows that the Krull dimension of R is at least 1. These two facts together yield that $p \in R$ is not nilpotent and hence there is a prime ideal I of R with $p \notin I$, and thus the fraction field of R/I is a finite extension E' of $\mathbb{Q}_p$. Thus the map $R \to R/I \hookrightarrow E'$ produces the desired lifting.

Remark. One may note that for $q \neq p$, $\tilde{\rho}q|\{I_q\}$ is non-trivial and (up to twist) unipotent, one initially gets a lift ρ such that $\rho|_{I_q}$ is (up to twist) unipotent but may in fact be trivial. One checks this cannot be the case by using a purity argument combined with the potential modularity of $\tilde{\rho}$. The corresponding local deformation ring at q is the one defined by Ramakrishna for his auxiliary primes (see [57]).

6. Applications of the S-conjecture

6.1 Modularity of compatible systems and rank 2 motives over $\mathbb{Q}$

The S-conjecture implies modularity of many motives of rank 2 over $\mathbb{Q}$. We prefer to formulate this implication for compatible systems instead.

We fix embeddings ι_p, ι_∞ of $\overline{\mathbb{Q}}$ in its completions $\overline{\mathbb{Q}_p}$ and $\mathbb{C}$ for each prime p.

Definition 6.1. *For a number field E, an E-rational, 2-dimensional compatible system of representations (ρ_λ) of $G_{\mathbb{Q}}$ is the data consisting of:*

(i) *for each finite place λ of E, $\rho_\lambda : G_{\mathbb{Q}} \to \mathrm{GL}_2(E_\lambda)$ is a continuous, absolutely irreducible representation of G_F,*
(ii) *there is a finite set of places S of $\mathbb{Q}$, such that for all finite places $q \notin S$ of $\mathbb{Q}$, if λ is a place of E of residue characteristic different from q, ρ_λ is unramified at q; further the characteristic polynomial $f_q(X)$ of $\rho_\lambda(\mathrm{Frob}_q)$ is in $E[X] \hookrightarrow E_\lambda[X]$ and independent of λ.*
(iii) *For almost all places λ of E above places ℓ of $\mathbb{Q}$, $\rho_\lambda|_{D_\ell}$ is crystalline at ℓ of integral Hodge-Tate weights (a, b) with $a \geq b$ that are independent of ℓ, and the (prime to ℓ) conductor $N(\rho_\lambda)$ of ρ_λ is bounded independently of λ.*

When $a \neq b$, then the system is said to be regular, otherwise irregular. We say that a compatible system is odd, if for one λ, and hence all λ, $\rho_\lambda(c)$ is not a scalar with $c \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ a complex conjugation.

By abusing notation, for a prime ℓ we denote by ρ_ℓ the ℓ-adic representation ρ_λ of the compatible system (ρ_λ) for λ the place above ℓ we have fixed. We denote by $\bar{\rho}_\ell$ the semisimple residual representation that arises from ρ_ℓ.

Compatible systems are expected to have the property that for all but finitely many λ, $N(\bar{\rho}_\lambda), k(\bar{\rho}_\lambda)$ are constant (i.e. independent of λ). The variability of these invariants for finitely many λ is exploited in the proof of Theorem 3.1.

As mentioned in Section 3 compatible systems play a key role in the proof of the S-conjecture. On the other hand the S-conjecture also has some iplications for 2-dimensional compatible systems. The following corollary of the S-conjecture is formulated in [51].

Theorem 6.2.

(i) *A regular two-dimensional compatible system that is odd arises up to twist from a newform of weight ≥ 2.*
(ii) *An irregular two-dimensional compatible system that is odd arises up to twist from a newform of weight* 1.

This strengthens a result of Taylor (see Theorem 5.6 of [78]). The two parts of the theorem are meant to indicate differences in the argument which relates to the fact that weight ≥ 2 forms are cohomological, while weight one forms are not. Note that irreducible motives of rank 2 over $\mathbb{Q}$ are expected to give rise to compatible systems of Galois representations as in (i) when they are regular (i.e. their Hodge-type is $(a, b), (b, a)$ for $a \neq b$).

The proofs are similar to the argument used by Serre in [66, Sections 4.6, 4.8] to deduce modularity of elliptic curves over $\mathbb{Q}$ from his conjecture, except that in (ii) we also have to make use of an observation of [45].

In both cases it is easy to see that $\bar{\rho}_\lambda$ is irreducible for almost all λ using the fact that the conductor of ρ_λ is bounded independently of λ and the Hodge-Tate weights are fixed.

Firstly after twisting we may assume that the Hodge-Tate numbers (a, b) of the compatible system are such that $b = 0$ and $a \geq 0$.

In the case of (i), when $a > 0$, we see that Serre's conjectures apply to $\bar{\rho}_\lambda$ for infinitely many λ and that these arise from a fixed newform $f \in S_k(\Gamma_1(N), \mathcal{O})$ for some integers k, N with $k > 1, N > 4$, and with $\mathcal{O}$ the ring of integers of a number field E. From this it follows, comparing the characteristic polynomials of Frobenii that arise from the compatible system attached to f and those attached to (ρ_λ), that (ρ_λ) arises from f.

In the case of (ii), when $a = b = 0$, the complication is that mod ℓ weight 1 forms need not lift to characteristic 0. The argument to circumvent this, gives a way of going from results about "regular" Galois representations to "irregular" ones and uses Proposition 4.1 (1).

By a theorem of Sen, and Fontaine, for all but finitely many λ, ρ_λ is unramified at $\ell(\lambda)$ the residue characteristic of λ. (See [65, Corollary 1] which proves the key result that $\rho_\lambda(I_{\ell(\lambda)})$ has finite image assuming only that ρ_λ is Hodge-Tate of weight 0.)

From this using the S-conjecture in its orginal formulation we can at first only conclude that $\bar{\rho}_\lambda$ for almost all λ arises from $S_{\ell(\lambda)}(\Gamma_1(N), \mathcal{O})$ $(N > 4)$ where $\ell(\lambda)$ is the residue charateristic of the residue field arising from λ. We cannot conclude the proof of the theorem at this juncture as the dimensions of $S_{\ell(\lambda)}(\Gamma_1(N))$ tend to infinity.

But using known cases of the characteristic p form of Serre's conjecture, in particular result of Gross, Coleman-Voloch [35,15] (this is the "non-formal" step of this extra argument in the irregular case) we get $\bar{\rho}_\lambda$ arises from a much smaller space $S_1(\Gamma_1(N), \mathbb{F}_\lambda)$ which by mutiplication by the Hasse invariant (and Proposition 4.1 (2)) we may identify to a subspace of $S_{\ell(\lambda)}(\Gamma_1(N), \mathcal{O}) \otimes \mathbb{F}_\lambda$. Here $\mathbb{F}_\lambda$ is the residue field of λ. The dimensions of $S_1(\Gamma_1(N), \mathbb{F}_\lambda)$ are bounded independently of ℓ, but this is again not quite enough to conclude the proof. We need to observe further that after throwing out a sporadic finite set of primes using Proposition 4.1(1) that $\bar{\rho}_\lambda$ arises from $S_1(\Gamma_1(N), \mathbb{C})$. Now we may conclude the proof of (ii) by arguing exactly as in the proof of (i).

6.1.1 Abelian varieties of GL_2-type and Artin's conjecture

Definition 6.3. *An abelian variety A over $\mathbb{Q}$ is said to be of GL_2-type if it is simple, and if there is a number field L such that $[L : \mathbb{Q}] = \dim(A)$, and an order $\mathcal{O}$ of L such that $\mathcal{O} \hookrightarrow \mathrm{End}_{\mathbb{Q}}(A)$.*

A corollary of part (i) of Theorem 6.2 is the generalised Shimura-Taniyama-Weil conjecture:

Corollary 6.4. *For an abelian variety A over $\mathbb{Q}$ of GL_2-type with conductor N, there is a non-constant morphism $\pi : X_1(N) \to A$ defined over $\mathbb{Q}$.*

The corollary follows from the part (i) of Theorem 6.2 (in the case of $|a - b| = 1$) and Faltings' isogeny theorem. For $|a-b| > 1$, we do not seem to be able to make a "motivic" statement for lack of a Tate conjecture for modular forms of weight $k > 2$ (see [90]). The reader may consult [5,6] for the Tate conjecture for the Scholl motive.

Part (ii) of Theorem 6.2 yields Artin's conjecture for 2-dimensional, odd, irreducible representations $\rho_{\mathbb{C}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ as an Artin representation $\rho_{\mathbb{C}}$ gives rise to a compatible system of the type in part (ii). It is not implied by Artin's conjecture as we do not know *a priori* that a compatible system as in (ii) arises after twisting from an Artin representation. If in part (ii) we omit the condition of *oddness*, then one would still expect that some twist of the compatible system arises from an Artin representation, but we have no clue of how one would prove this.

6.2 Descent for Hilbert modular forms over non-solvable extensions $F/\mathbb{Q}$

Hida and Wintenberger have pointed out the following consequence which follows from the S-conjecture.

Theorem 6.5. *Let $F/\mathbb{Q}$ be a totally real, Galois extension and let $G = \mathrm{Gal}(F/\mathbb{Q})$. Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ that is holomorphic of weight $(k, \ldots, k)$ at infinity for k an integer ≥ 1, and assume that $\pi^\sigma \simeq \pi$ for $\sigma \in G$. Then there is a character ψ of finite order of G_F, such that $\pi \otimes \psi$ is the base change of a cuspidal automorphic representation Π of $G_{\mathbb{Q}}$ that is holomorphic of weight k at infinity.*

We know that π gives rise to a compatible system of representations (ρ_λ) that are all irreducible and residually irreducible for almost all λ. Consider ρ_p for $p >> 0$ and at which $F/\mathbb{Q}$ is unramified and such that ρ_p is crystalline at all places above p. Using a well-known theorem of Tate (see Theorem 4 of [71]), one easily sees that there is a representation $\rho'_p : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$ such that $\rho'_p|_{G_F} = \rho_p \otimes \psi$ for some character $\psi : G_F \to \overline{\mathbb{Q}_p}^*$. Methods of Wintenberger [87] show that ρ'_p may be chosen to be crystalline at p of weight k and thus ψ is of finite order and unramified at places above p. We know that ρ'_p is modular by using the S-conjecture and modularity lifting theorems in [20].

6.3 A couple of questions about Artin representations

This might be a good place to pose a couple of questions, not directly related but part of the circle of ideas of this note.

1. Consider K a function field of characteristic p and a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_K)$ with finite order central character. Then from the work of Drinfeld and Lafforgue we know how to attach an ℓ-adic ($\ell \neq p$) representation ρ to π that is compatible with local Langlands correspondence. For almost all places v of K, ρ is unramified at v and $\rho(\mathrm{Frob}_v)$ has eigenvalues that are algebraic numbers and of absolute value 1 under all embeddings of $\overline{\mathbb{Q}_\ell}$ into $\mathbb{C}$ and are integral outside p. If these are integral at p then it is not hard to see that ρ is an Artin representation. Is there a better criterion in terms of π that predicts when ρ is Artin? In the number field case the Langlands parameter at the archimedean places attached to a cuspidal automorphic representation of say $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ is expected to predict this. One is asking for a method of attaching Hodge numbers to cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_K)$ with K a function field of characteristic p, or of the possibility of a p-adic representation associated to π which will reveal if the representation π is Artin or not.
2. There is an entire spectrum of forms between those which are Artin to those which are cohomological when working with say groups like $\mathrm{GSp}(4)$. Can we come up with arguments to deal with the difficulty of moving across this spectrum? In general there will be an additional complication that while Artin representations automatically give rise to compatible systems the intermediate ranges of the spectrum will not have this property. For example given an irreducible finite flat group scheme over $\mathbb{Z}$ that is annihilated by p and of rank $n > 2$ one does not know that it arises from an abelian scheme over $\mathbb{Z}$, or as the reduction of the p-adic representation arising from a p-divisible group over $\mathbb{Z}$ (although this might be accessible), or that it is part of a compatible system of mod ℓ representations of $G_\mathbb{Q}$ in the sense of [49].

7. What next? Modularity and motives with good reduction

One is reminded by the techniques used in the proof of the S-conjecture of a remark of Archimedes quoted by Pappus of Alexandria: *Give me but one firm spot on which to stand, and I will move the earth.* (The Oxford Dictionary of Quotations, Second Edition, Oxford University Press, London, 1953, p. 14.) The purpose of this section is to speculate rather idly about the existence of such firm spots in the quagmire of higher-dimensional representations of $G_\mathbb{Q}$.

The proof of the S-conjecture in retrospect can be viewed as a method to exploit an accident which occurs in a few different guises:

1. (Fontaine, Abrashkin) There are no non-zero abelian varieties over $\mathrm{Spec}(\mathbb{Z})$.
2. (Tate, Serre) There are no irreducible representations $\bar{\rho} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}})$ with $\overline{\mathbb{F}}$ the algebraic closure of either $\mathbb{F}_2$ or $\mathbb{F}_3$ that are unramified outside 2 and 3 respectively.
3. $H^*_{\mathrm{cusp}}(\mathrm{SL}_2(\mathbb{Z}), \mathbb{R}) = 0$: i.e., $\mathrm{SL}_2(\mathbb{Z})$ has no cuspidal cohomology with constant real coefficients.

These accidents allow one to prove Serre's conjecture for very small invariants $(p, k(\bar{\rho}), N(\bar{\rho}))$ attached to $\bar{\rho}$ (for example $p = 2, N(\bar{\rho}) = 1, k(\bar{\rho}) = 2$).

Given these accidents, there is a method developed in the course of proving Theorem 3.1 to exploit them to prove Serre's conjecture for all other triples of invariants by linking different invariants by a series of *linked* compatible systems as defined below.

Definition 7.1. *Let E be a number field. We say that two E-rational, compatible systems of representations of $G_{\mathbb{Q}}$ are* **linked** *(at λ) if for some finite place λ of E the semisimplifications of the corresponding residual mod λ representations arising from the two systems are isomorphic up to a twist by a (one-dimensional) character of $G_{\mathbb{Q}}$.*

The utility of the definition is that the modularity lifting techniques pioneered by Wiles often allow one to prove if one of two linked compatible systems is modular, then so is the other.

When thinking of generalisations of Serre's conjecture that concerns representations $\bar{\rho} : G_F \to \mathrm{GL}_n(\mathbb{F})$ in the case of $F = \mathbb{Q}, n = 2$ one may allow more general F or consider larger values of n,

If we allow F to vary, the problem we face is that the accidents above do not persist even for a general real quadratic field F.

Curiously enough the other direction of fixing $F = \mathbb{Q}$ but varying n seems more promising.

If we consider higher dimensional representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, the accidents persist for a while:

(a) (Fontaine [29]) There is no 7-adic n-dimensional ($n > 1$) representation of $G_{\mathbb{Q}}$ that is unramified outside 7, and crystalline at 7 with Hodge-Tate weights $\in [0, 3]$, that is irreducible.
(b) (Fermigier, Miller [27,56]) $H^*_{\mathrm{cusp}}(\mathrm{SL}_n(\mathbb{Z}), \mathbb{R}) = 0$ for $n \leq 23$: i.e., $\mathrm{SL}_n(\mathbb{Z})$ has no cuspidal cohomology with constant real coefficients for $n \leq 23$.

The n-dimensional p-adic representations of $G_{\mathbb{Q}}$ which are expected to be attached to cusp forms that contribute to $H^*_{\mathrm{cusp}}(\mathrm{SL}_n(\mathbb{Z}), \mathbb{R})$ are supposed to have the property that they are irreducible, unramified outside p, and crystalline at p with Hodge-Tate numbers $0, 1, \ldots, n-1$.

Conjecture 1. *For $n \leq 23$ there are no non-trivial (which one has to rule out only when $n = 1$), irreducible n-dimensional p-adic representations $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{Q}_p})$ unramified outside p and crystalline at p with Hodge-Tate weights* $0, 1, \ldots, n-1$.

This is proved for $n = 2$ in [21] and [87]. We can analogously guess that there are no irreducible, rank $n \leq 23$ motives over $\mathbb{Q}$, with good reduction everywhere and with Hodge type $(n-1, 0), (n-2, 1), \ldots, (0, n-1)$.

One can be yet more ambitious and ask if this can be true for all n: then it is not even clear what to expect. Even the case of small n will require new ideas as $n = 4$, even for small primes p, is essentially the limit of the proof in [29].

One can also ask a similar question about existence of cusp forms on $\mathrm{SL}_n(\mathbb{Z})$ with infinitesimal character that of the trivial representation.

If by some miracle the conjecture is true for all n, it would be a striking generalisation of the Hermite-Minkowski theorem that $\mathbb{Q}$ has no non-trivial unramified extensions. Note that the latter implies the statement above for $n = 1$ as it implies $G_{\mathbb{Q}}$ has no non-trivial characters that are unramified at all finite places. In fact by the results of Sen and Fontaine, and the Hermite-Minkowski theorem, one knows that if $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{Q}_p})$ is a continuous representation, unramified outside p, and crystalline at p with Hodge-Tate weights all 0, then ρ is unramified everywhere and hence the trivial representation.

One might hope that Fontaine's result in [29] gives a toe-hold to try and extend the methods of the proof of Theorem 3.1 to certain representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ of dimension up to 4. For instance consider the following standard conjecture (we thank Ash for correcting a misstatement of this in a talk at Boston University):

Conjecture 2. *Let* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSp}_4(\mathbb{F})$ *be an irreducible representation that is* odd *(i.e., the similitude character is odd). Then $\bar{\rho}$ arises from an automorphic representation of* $\mathrm{GSp}_4(\mathbb{A})$.

There is a lot work to be done in order to fulfill this hope which will for instance involve developments in the p-adic Langlands program and the modularity lifting machinery, and extensions of the results of [36].

The relevance of the lowest Hodge-Tate (HT) weights $(0, 1, \ldots, n-1)$ is that the method in the classical case works via constructing liftings, which rely on results (presentation results for global deformation rings, $R = \mathbb{T}$) that work well only when the lifts we are trying to construct have prescribed integral HT weights that are *distinct*. Thus although [28] proves results about n-dimensional p-adic representations of $G_{\mathbb{Q}}$ for small primes (p odd and ≤ 17) that are unramified outside p and crystalline at p with HT weights $\in \{0, 1\}$ for any dimension, these results are not directly useful for $n > 2$

We quote following result of Fontaine and Abrashkin ([29] and [1]):

Theorem 7.2.

(i) *Consider a smooth projective variety X over $\mathbb{Q}$ with good reduction everywhere. Then $H^i(X.\Omega^j) = 0$ for i, j non-negative integers and $i + j \leq 3$,except when $i = j$.*
(ii) *Consider a smooth projective variety X over $\mathbb{Q}(i), \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{5})$ with good reduction everywhere. Then $H^i(X, \Omega^j) = 0$ for i, j non-negative integers and $i + j \leq 2$,except when $i = j$.*

Question 7.3. *Can one classify smooth projective surfaces over $\mathbb{Z}$? Are they always unirational?*

The results of [1] together with the methods in [51,52] may be useful in attacking the conjectures of [2] on a 3-dimensional version of the Shimura-Taniyama conjecture when the K there is $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$.

We remark that the method of killing ramification (see last section of [50]) allows one to relate, *via* linked compatible systems, motives with bad reduction properties to those with good reduction. From the geometric viewpoint this is a little exotic as the weight of the motives that get related are very different.

In the speculative vein of this section one might ask for a generalisation of the beautiful result of [28]. We are asking if there is a common generalisation of the results of [28] and those of Faltings in [25] proving Shafarevich's conjecture.

Question 7.4. *Given a number field K and a finite set of finite places S of K, are there only finitely many absolutely simple abelian varieties over K with good reduction outside S and semistable reduction at S?*

We could consider a related question for Artin representations which conjecturally corresponds to the case when at places above p the p-adic representation of G_K is crystalline and all its Hodge-Tate weights are 0. Serre observed that, if the p-class field tower of a number field K is infinite, then the group G_K has irreducible, complex, everywhere unramified representations of degree p^n, for an infinity of values of n.

Our question is clearly related to asking for a generalisation of the finiteness conjectures in I.3 of [32] about p-adic representations of G_K where instead of fixing the Hodge-Tate weights we only bound them, and do not bound their multiplicities, and we consider only representations which are irreducible on restriction to all open subgroups of G_K.

8. Counting Galois representations: asymptotics

It is a simple corollary of Theorem 3.1 that there are only finitely many semisimple 2-dimensional mod p representations of $G_{\mathbb{Q}}$ of bounded (prime-to-p

Artin) conductor. It will be of interest to get quantitative refinements of this. It is easy to see after Theorem 3.1 that for a fixed prime p the number $N(2,p)$ of isomorphism classes of continuous semisimple odd representations $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}_p})$ that are unramified outside p is bounded above by Cp^3 for a constant C independent of p. This is seen as by Theorem 3.1 we are counting the number of distinct level 1 mod p Hecke eigensystems, and then using formulas for the dimension of $S_k(\mathrm{SL}_2(\mathbb{Z}))$, and the fact that up to twist all mod p forms have weight $\leq p+1$, we are done. We would guess that $N(2,p)$ is asymptotic to $\frac{p^3}{48}$ with p.

Can we prove that there is a non-zero constant c, independent of p, such that this number is bounded below by cp^3? We can rather easily see by looking at reducible semisimple representations unramified outside p that there is a non-zero constant c (independent of p) such that this number is bounded below by cp^2.

Serre has indicated in an e-mail to the author an argument that bounds the number of such irreducible representations below by cp^2 for a non-zero constant c. We paraphrase his message. Suppose we have some k, with $14 < k < p$, k even, which is p-regular, i.e., such that the numerator of the kth Bernoulli number B_k is not divisible by p. Choose a newform of that weight, and its corresponding Galois representation in characteristic p that is unramified outside p; lets call it r. It is standard that r will be irreducible.

Hence, each p-regular k gives us at least one irreducible r (although there should really be many more!). Claim: the number of such k is at least $\frac{p}{4}+o(p)$. Indeed the total number of the even k's is $\frac{p}{2}+O(1)$. Call t the number of the irregular ones. By a known formula, the factor $h^-(p)$ of the p-th cyclotomic field is divisible by p^t. But Carlitz has given an upper bound for $h^-(p)$ which is roughly of the size of $p^{\frac{p}{4}}$ (see [61]). This proves the claim.

The representations just constructed are distinct. We may twist them by powers of the cyclotomic character. The resulting twists of a given r are almost all distinct: at most two of them can be isomorphic. We thus get about $\frac{p^2}{8}$ distinct irreducible representations unramified outside p.

For large enough k, the maximal ideals m of the Hecke algebra (where for simplicity we consider only the Hecke operators T_n for $(n,p)=1$) which are in the support $S_k(\mathrm{SL}_2(\mathbb{Z}),\mathbb{F})$ "depend" only on k mod $p-1$. Fixing a m we may ask if $\frac{\dim_{\mathbb{F}}(S_k(\mathrm{SL}_2(\mathbb{Z}),\mathbb{F})_{\mathsf{m}})}{\dim_{\mathbb{F}}(S_k(\mathrm{SL}_2(\mathbb{Z}),\mathbb{F}))}$ tends to a limit as k runs through all integers with $\overline{\chi}_p{}^{k-1} = \det(\bar{\rho}_{\mathsf{m}})$, and if the limit is independent of the m's with $\det(\bar{\rho}_{\mathsf{m}}) = \overline{\chi}_p{}^{k-1}$ for such k's.

9. Erratum to [44]

- p. 559, Corollary 1.4 (ii): the condition that k is even may be suppressed.
- p. 560, l. 15: delete the word "because".

- p. 561: Moret and Bailly should be Moret-Bailly.
- p. 563: the reference [46] should be [47].
- p. 568: in 3.2(ii) no indentation.
- p. 570: the equation $\beta^2 - \beta\gamma(\chi'(\tau) - 1) - \psi$ should be $\beta^2 + \beta\gamma(\chi'(\tau) - 1) - \psi(\sigma_q)$.
- p. 571: in the item $(B_{\mathbb{Q}})$ delete "(i.e. the condition in Section 3.2(i))"
- p. 573: delete the last sentence before Section 4.2, and the parentheses in the first sentence of Section 4.2.
- p. 578: add "of dimension 2" at the end of the second paragraph.
- p. 580: the numbering (1) and (2) in the proof of Theorem 6.1 could be misleading: as said there it refers to cases (1) and (3), and (2) of the theorem respectively.
- p. 580, Lemma 6.2 (i): replace $K = \mathbb{Q}(\mu_p)$ by $K = \mathbb{Q}(\sqrt{(-1)^{\frac{p-1}{2}} p})$.
- p. 584: in the third paragraph l should be ℓ, in the first sentence of (2) p should be P. The indentation after the first paragraph of (2) might be misleading as the rest of the section is all in subcase (2).

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References

[1] Victor Abrashkin, Modular representations of the Galois group of a local field and a generalization of a conjecture of Shafarevich, *Izv. Akad. Nauk SSSR Ser. Mat.*, **53** (1989) no. 6, 1135–1182, 1337; translation in Math. USSR-Izv. 35 (1990) no. 3, 469–518.

[2] Don Blasius, A rank three generalisation of the Shimura and Taniyama conjecture, preprint.

[3] Gebhard Böckle, A local-to-global principle for deformations of Galois representations, *J. Reine Angew. Math.*, **509** (1999) 199–236.

[4] Gebhard Böckle, Presentations of universal deformation rings. preprint.

[5] Alex Brown and Eknath Ghate, Endomorphism algebras of motives attached to elliptic modular forms, *Ann. Inst. Fourier (Grenoble)*, **53** (2003) no. 6, 1615–1676.

[6] E. Ghate, E. González-Jiménez and J. Quer, On the Brauer class of modular endomorphism algebras, *Int. Math. Res. Not.*, (2005) no. 12, 701–723.
[7] Kevin Buzzard, Fred Diamond, and Frazer Jarvis, On Serre's conjecture for mod ℓ Galois representations over totally real number fields, preprint (2005).
[8] Sharon Brueggeman, The nonexistence of certain Galois extensions unramified outside 5, *J. Number Theory*, **75(1)** (1999) 47–52.
[9] K. Buzzard, M. Dickinson, N. Shepherd-Barron, R. Taylor, On icosahedral Artin representations, *Duke Math. J.*, **109** (2001) 283–318.
[10] Armand Brumer and Kenneth Kramer, Non-existence of certain semistable abelian varieties, *Manuscripta Math.*, **106(3)** (2001) 291–304.
[11] Kevin Buzzard, On level-lowering for mod 2 representations, *Math. Res. Lett.*, **7** (2000) no. 1, 95–110.
[12] Henri Carayol, Sur les representations galoisiennes modulo ℓ attaches aux formes modulaires, *Duke Math. J.*, **59** (1989), no. 3, 785–801.
[13] Henri Carayol, Formes modulaires et représentations galoisiennes à valeurs dans un anneau local complet, In p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991), volume 165 of *Contemp. Math.*, pages 213–237. Amer. Math. Soc. (1994).
[14] L. Clozel, M. Harris, R. Taylor, Automorphy for some l-adic lifts of automorphic mod l representations, preprint.
[15] Robert Coleman, Jos Felipe Voloch, Companion forms and Kodaira-Spencer theory, *Invent. Math.*, **110** (1992) no. 2, 263–281.
[16] Pierre Colmez, Une correspondance de Langlands locale p-adique pour les representations semi-stables de dimension 2, preprint (2004).
[17] Fred Diamond, The Taylor-Wiles construction and multiplicity one, *Invent. Math.*, **128** (1997) no. 2, 379–391.
[18] Fred Diamond and Richard Taylor, Lifting modular mod ℓ representations, *Duke Math. J.* **74** (1994) no. 2, 253–269.
[19] Henri Darmon, Fred Diamond, and Richard Taylor, Fermat's last theorem, In Current developments in mathematics, 1995 (Cambridge, MA), pages 1–154. Internat. Press, Cambridge, MA (1994).
[20] Fred Diamond, Matthias Flach, Li Guo, The Tamagawa number conjecture of adjoint motives of modular forms, *Ann. Sci. École Norm. Sup. (4)* **37** (2004) no. 5, 663–727.
[21] Luis Dieulefait, Existence of families of Galois representations and new cases of the Fontaine-Mazur conjecture, *J. Reine Angew. Math.* **577** (2004) 147–151.
[22] Bas Edixhoven, The weight in Serre's conjectures on modular forms, *Invent. Math.*, **109** (1992) no. 3, 563–594.
[23] Bas Edixhoven, Serre's conjecture, Modular forms and Fermat's last theorem (Boston, MA, 1995), 209–242, Springer, New York, (1997).
[24] Bas Edixhoven, Comparison of integral structures on spaces of modular forms of weight two, and computation of spaces of forms mod 2 of weight one. With appendix A (in French) by J.-F. Mestre and appendix B by Gabor Wiese, *J. Inst. Math. Jussieu*, **5** (2006) no. 1, 1–34.
[25] Gerd Faltings, Endlichkeitssätze für abelsche Varietäten über Zahlkörpern, *Invent. Math.*, **73** (1983) no. 3, 349–366.
[26] Kazuhiro Fujiwara, Deformation rings and Hecke algebras for totally real fields, preprint (1999).
[27] Stéfane Fermigier, Annulation de la cohomologie cuspidale de sous-groupes de congruence de $\mathrm{GL}_n(\mathbb{Z})$, *Math. Ann.*, **306** (1996) no. 2, 247–256.
[28] Jean-Marc Fontaine, Il n'y a pas de variété abélienne sur **Z**, *Invent. Math.*, **81(3)** (1985) 515–538.
[29] Jean-Marc Fontaine, Schémas propres et lisses sur $\mathbb{Z}$, Proceedings of the Indo-French Conference on Geometry (Bombay, 1989), 43–56, Hindustan Book Agency, Delhi (1993).

[30] Jean-Marc Fontaine, Luc Illusie, p-adic periods: a survey, Proceedings of the Indo-French Conference on Geometry (Bombay, 1989), 57–93, Hindustan Book Agency, Delhi (1993).
[31] Jean-Marc Fontaine and Guy Laffaille, Construction de représentations p-adiques, *Ann. Sci. École Norm. Sup. (4)*, **15** (1982) 4, 547–608.
[32] Jean-Marc Fontaine, Barry Mazur, Geometric Galois Representations, in Elliptic Curves, Modular Forms, and Fermat's Last Theorem, Coates, J. and Yau, S.T. ed, Internat. Press, Cambridge, MA (1995).
[33] Toby Gee, Companion forms over totally real fields, II, preprint.
[34] Toby Gee, Automorphic lifts with prescribed types, preprint.
[35] Benedict Gross, A tameness criterion for Galois representations associated to modular forms (mod p), *Duke Math. J.*, **61** (1990) no. 2, 445–517.
[36] M. Harris, N. Shepherd-Barron and R. Taylor, A family of Calabi-Yau varieties and potential automorphy, preprint.
[37] Nicholas Katz, p-adic properties of modular schemes and modular forms, Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 69–190. Lecture Notes in Mathematics, Vol. 350, Springer, Berlin (1973).
[38] Nicholas Katz, A result on modular forms in characteristic p, *SLNM*, **601** (1976) 53–61.
[39] Chandrashekhar Khare and Ravi Ramakrishna, Finiteness of Selmer groups and deformation rings, *Invent. Math.*, **154(1)** (2003) 179–198.
[40] Mark Kisin, Moduli of finite flat group schemes, and modularity, preprint (2004).
[41] Mark Kisin, The Fontaine-Mazur conjecture for GL_2, preprint (2006).
[42] Mark Kisin, Modularity of Barsotti-Tate representations, preprint, notes for the *Current Developments in Mathematics* conference (2005).
[43] Mark Kisin, Potentially semistable deformation rings, preprint.
[44] C. Khare, Serre's modularity conjecture: the level one case, *Duke Math. J.*, **134(3)** (2006) 557–589.
[45] C. Khare, Remarks on mod p forms of weight one, International Math. Res. Notices, vol. 3 (1997), 127–133. (Corrigendum: IMRN 1999, no. 18, pg. 1029.)
[46] C. Khare, A local analysis of congruences in the (p, p) case: Part II, *Inventiones Mathematicae*, **143** (2001) 129–155.
[47] C. Khare, Serre's modularity conjecture: a survey of the level one case, to appear in Proceedings of the London Math Soc conference on L-functions and Galois representations, Durham (2004), ed. D. Burns, K. Buzzard, J. Nekovar.
[48] C. Khare, D. Prasad, Extending local representations to global representations, *Journal of Mathematics of Kyoto University* **36** (1996) 471–480.
[49] C. Khare, Compatible systems of mod p Galois representations, C. R. Acad. Sci., Paris (1996), t. 323, Série I, 117–120.
[50] C. Khare and Jean-Pierre Wintenberger, On Serre's reciprocity conjecture for 2-dimensional mod p representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, preprint available at http://xxx.lanl.gov/abs/math.NT/0412076.
[51] C. Khare and Jean-Pierre Wintenberger, Serre's modularity conjecture (I), preprint available at http://www.math.utah.edu/~shekhar/papers.html.
[52] C. Khare and Jean-Pierre Wintenberger, Serre's modularity conjecture (II), preprint available at http://www.math.utah.edu/~shekhar/papers.html.
[53] Robert Langlands, Base change for GL_2, Annals of Math. Series, Princeton University Press (1980).
[54] Barry Mazur, Deforming Galois representations, Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987), 385–437, Math. Sci. Res. Inst. Publ., 16, Springer, New York (1989).
[55] Barry Mazur, An introduction to the deformation theory of Galois representations, In Modular forms and Fermat's last theorem (Boston, MA, 1995), pages 243–311. Springer, New York (1997).

[56] Stephen Miller, Spectral and cohomological applications of the Rankin-Selberg method, *Internat. Math. Res. Notices*, (1996) no. 1, 15–26.

[57] Ravi Ramakrishna, Deforming Galois representations and the conjectures of Serre and Fontaine-Mazur, *Ann. of Math. (2)*, **156** (2002) no. 1, 115–154.

[58] K. A. Ribet, Report on mod ℓ representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, in Motives, Proc. Sympos. Pure Math., 55, Part 2, Amer. Math. Soc., 639–676.

[59] K. A. Ribet, On modular representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ arising from modular forms, *Invent. Math.*, **100(2)** (1990) 431–476.

[60] K. A. Ribet, Congruence relations between modular forms, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 503–514, PWN, Warsaw (1984).

[61] K. A. Ribet, A modular construction of unramified p-extensions of $Q(_p)$, *Invent. Math.*, **34** (1976) no. 3, 151–162.

[62] David Savitt, Modularity of some potentially Barsotti-Tate Galois representations, *Compositio Math.*, **140** (2004) 31–63.

[63] René Schoof, Abelian varieties over cyclotomic fields with good reduction everywhere, *Math. Ann.*, **325** (2003) no. 3, 413–448.

[64] René Schoof, Abelian varieties over **Q** with bad reduction in one prime only, *Compos. Math.*, **141** (2005) no. 4, 847–868.

[65] Shankar Sen, Lie algebras of Galois groups arising from Hodge-Tate modules, *Annals of Math.*, **97** (1973) 160–170.

[66] Jean-Pierre Serre, Sur les représentations modulaires de degré 2 de $Gal(\bar{Q}/Q)$, *Duke Mathematical Journal*, **54** (1987) 179–230.

[67] Jean-Pierre Serre, Une interprétation des congruences relatives à la fonction τ de Ramanujan, In Séminaire Delange-Pisot-Poitou: 1967/68, Théorie des Nombres, Fasc. 1, Exp. 14, page 17. Secrétariat mathématique, Paris (1969).

[68] Jean-Pierre Serre, Formes modulaires et fonctions zêta p-adiques, In Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972), pages 191–268. Lecture Notes in Math., Vol. 350, Springer, Berlin (1973).

[69] Jean-Pierre Serre, Congruences et formes modulaires [d'aprs H. P. F. Swinnerton-Dyer], Séminaire Bourbaki, 24e anne (1971/1972), Exp. No. 416, pp. 319–338. Lecture Notes in Math., Vol. 317, Springer, Berlin (1973).

[70] Jean-Pierre Serre, Œuvres. Vol. III, Springer-Verlag, Berlin, (1986) 1972–1984.

[71] Jean-Pierre Serre, Modular forms of weight one and Galois representations, pgs. 292–367, Œuvres. Vol. III, Springer-Verlag, Berlin (1986).

[72] Jean-Pierre Serre, Abelian ℓ-adic representations and elliptic curves, Advanced Book Classics. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA (1989) xxiv+184 pp.

[73] Christopher Skinner and Andrew Wiles, Base change and a problem of Serre, *Duke Math. J.*, **107(1)** (2001) 15–25.

[74] Christopher Skinner and Andrew Wiles, Residually reducible representations and modular forms, *Inst. Hautes Études Sci. Publ. Math.*, **(89)** (2000, 1999) 5–126.

[75] Christopher Skinner and Andrew Wiles, Nearly ordinary deformations of irreducible residual representations, *Ann. Fac. Sci. Toulouse Math. (6)*, **10(1)** (2001) 185–215.

[76] H. P. F. Swinnerton-Dyer, On ℓ-adic representations and congruences for coefficients of modular forms, Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972), pp. 1–55. Lecture Notes in Math., Vol. 350, Springer, Berlin (1973).

[77] John Tate, The non-existence of certain Galois extensions of **Q** unramified outside 2, *Arithmetic geometry (Tempe, AZ, 1993)*, Contemp. Math., 174, 153–156, Amer. Math. Soc., (1994).

[78] Richard Taylor, Galois representations, Annales de la Faculte des Sciences de Toulouse 13 (2004), 73–119.

[79] Richard Taylor, Remarks on a conjecture of Fontaine and Mazur, *Journal de l'Institut de Math, de Jussieu*, **1** (2002), 1–19.
[80] Richard Taylor, On the meromorphic continuation of degree two L-functions, Preprint, pages 1–53 (2001).
[81] Richard Taylor, Andrew Wiles.*Ring-theoretic properties of certain Hecke Algebras*, *Annals of Mathematics (2)*, **141** (1995) no. 3, 553–572.
[82] Richard Taylor, Automorphy for some ℓ-adic lifts of automorphic mod ℓ representations, II preprint.
[83] Adrian Vasiu, Surjectivity criteria for p-adic representations, I, *Manuscripta Math.*, **112** (2003) no. 3, 325–355.
[84] Marie-France Vignéras, Représentations ℓ-modulaires d'un groupe reduct if p-adique avec $l \neq p$. Progress in Mathematics, 137. Birkhäuser Boston, Inc., Boston, MA (1996), xviii and 233 pp.
[85] Gabor Wiese, Dihedral Galois representations and Katz modular forms, Documenta Mathematica, Vol. 9 (2004) 123–133.
[86] Andrew Wiles, Modular Elliptic Curves and Fermat's Last Theorem, *Annals of Mathematics* **141** (1995) 443–551.
[87] Jean-Pierre Wintenberger, Sur les représentations p-adiques géométriques de conducteur 1 et de dimension 2 de $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, preprint available at http://xxx.lanl.gov/abs/math.NT/0406576.
[88] Jean-Pierre Wintenberger, La conjecture de modularité de Serre: le cas de conducteur 1 [d'après C. Khare] Seminaire Bourbaki, Novembre (2005).
[89] Jean-Pierre Wintenberger, On p-adic geometric representations of $G_{\mathbb{Q}}$, to appear in Documenta Mathematics, volume in honor of J. Coates.
[90] Takuya Yamauchi, $\mathbb{Q}$-motives and modular forms, preprint.