DISTORTION AND ℓ_1-HOMOLOGY

S. M. Gersten

Abstract. If H is a finitely generated subgroup of the finitely generated group G, then H is undistorted in G for their word metrics iff the induced homomorphism on the zero-dimensional ℓ_1-homology group is injective. The Mayer-Vietoris exact sequence for ℓ_1-homology is established and applications are given to hyperbolic groups and relative hyperbolicity. The notion of relative distortion of two maps is introduced and a homological criterion for it is established.

§1. Introduction.

A finitely generated subgroup H of the finitely generated group G is said to be undistorted for their respective word metrics (*a.k.a.* the inclusion $H < G$ is a quasi-isometric imbedding) if the distance between two elements of H, as measured in the word metric of G, is bilipschitz equivalent to the distance as measured in H. In [Ge3] it was established that a finitely generated subgroup H of a hyperbolic group G is quasi-isometrically imbedded in G iff the restriction map on ℓ_∞-cohomology $H^1_{(\infty)}(G,\mathbb{Z}) \to H^1_{(\infty)}(H,\mathbb{Z})$ is surjective. Since the cohomological characterization of hyperbolic groups proved in [Ge2], that a finitely presented group G is hyperbolic iff $H^2_{(\infty)}(G,\ell_\infty) = 0$, was converted into the homological criterion in [AG], that G is hyperbolic iff $\bar{H}^{(1)}_2(G,\mathbb{R}) = H^{(1)}_1(G,\mathbb{R}) = 0$, it was natural to ask whether there was a corresponding homological characterization for a general finitely generated subgroup H of a finitely generated group G to be undistorted for their word metrics. We shall prove in Corollary 3.3 that *if H is a finitely generated subgroup of the finitely generated group G, then the inclusion $H < G$ is undistorted for their word metrics iff the induced map $H^{(1)}_0(H,\mathbb{R}) \to H^{(1)}_0(G,\mathbb{R})$ is injective.* This result follows from a more general result Theorem 3.2 on graphs: *If Δ is a connected subgraph of the graph Γ, then the inclusion $\Delta \subset \Gamma$ does not distort their respective word metrics (i.e. the inclusion is a quasi-isometric imbedding) iff the induced map $H^{(1)}_0(\Delta,\mathbb{R}) \to H^{(1)}_0(\Gamma,\mathbb{R})$ is injective.*

In §4 we prove a generalization of Theorem 3.2 that handles the case when Δ is not connected. Of particular interest here is Theorem 4.7, which says, in the terminology of Definition 4.8, that to detect distortion one need only look at *unramified* boundaries.

In §5 we establish a Mayer-Vietoris exact sequence for ℓ_1-homology and consider applications to hyperbolic groups and relative hyperbolicity. In particular we show in Theorem 5.6 that *if $G = A *_C B$ (resp. $G = A*_C$) where G, A, and B are*

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hyperbolic and C is finitely presented (*resp. G and A are hyperbolic and C is finitely presented*), *then C is hyperbolic.*

In §6 we introduce the notion of relative distortion of two maps $f, g : \Delta \to \Gamma$ of graphs and prove a homological criterion, Theorem 6.3, that f be undistorted relative to g which generalizes the results of the preceding three sections.

Appendix A contains the proof of the theorem that *if $G = A *_C B$ where G is of type $FP(n)$ and both A and B are of type $FP(n-1)$ where $n \geq 2$ and where C is finitely generated, then C is of type $FP(n-1)$* (a similar result holds in the HNN case). Appendix B, written after the Pusan conference, contains a geometrical interpretation of the distortion function for a finitely generated subgroup of a finitely generated group, and Appendix C contains a cohomological criterion for a subgroup to be undistorted.

I spoke on some of the results of this paper at conferences on group theory in Mobile in March, 1998, and in Pusan in August, 1998. I wish to take this opportunity to thank the organizers of these conferences for their kind hospitality.

§2. The ℓ_1-homology and ℓ_∞-cohomology of a group.

A norm on an abelian group A is a function $|\ \ | : A \to \mathbb{R}$ satisfying $|-a| = |a|$, $|a + a'| \leq |a| + |a'|$, and $|a| \geq 0$ with $|a| = 0$ iff $a = 0$, for all $a, a' \in A$.

We recall that a group G is said to be of type F_n if there is a CW-complex X' of type $K(G, 1)$ with finite n-skeleton. For example, G is of type F_1 iff it is finitely generated and of type F_2 iff it is finitely presented.

If G is a group of type F_{n+1}, let X' be a CW-complex of type $K(G, 1)$ with finite $(n+1)$-skeleton and let X be the universal cover of X'. A summable i-chain f on X with values in A is a skew-symmetric function from the oriented i-cells of X with values in A (so $f(\bar{e}) = -f(e)$ where e is an i-cell and $\bar{e}$ is the same geometric i-cell with the opposite orientation[1]) such that $\sum_e |f(e)| < \infty$, where the sum is over all oriented i-cells e. It is convenient to think of the chain f as an infinite sum $\sum_{e \in \mathcal{O}} f(e)e$ where $\mathcal{O}$ is an orientation on the i-cells, that is $\mathcal{O}$ contains precisely one of the pair e, $\bar{e}$ for each oriented i-cell e. Then we define the ℓ_1-norm $|f|_1$ of f by

$$|f|_1 = \sum_{e \in \mathcal{O}} |f(e)|. \tag{2.1}$$

If $i \leq n+1$ then because X' has only finitely many i-cells, there is an upper bound on the ℓ_1-norms of the boundaries ∂e of i-cells e. This means that if $C_i^{(1)}(X, A)$ denotes the set of summable i-chains with values in A, then the function ∂ extends to a continuous homomorphism $\hat{\partial} : C_i^{(1)}(X, A) \to C_{i-1}^{(1)}(X, A)$. One checks that $\hat{\partial}^2 = 0$, so we have a chain complex defined in a range of degrees $i \leq n+1$. The ℓ_1-homology $H_i^{(1)}(X, A)$ is defined in the usual way as $Z_i^{(1)}(X, A)/B_i^{(1)}(X, A)$, where $Z_i^{(1)}(X, A)$ is the subgroup of summable i-cycles and where $B_i^{(1)}$ is the image of $\hat{\partial} : C_{i+1}^{(1)}(X, A) \to C_i^{(1)}(X, A)$. This makes sense for $i \leq n$.

Now we consider the ℓ_∞ cohomology groups. We define $C^i_{(\infty)}(X, A)$ to be the subgroup of cellular i-cochains h such that there is a number $M_h > 0$ with $|h(\sigma)| \leq M_h$ for all i-cells σ. It follows from the finiteness of ${X'}^{(n+1)}$ that the coboundary δh lies

[1] Precisely, $\bar{e}$ is obtained from e by precomposing the characteristic mapping of e with some

in $C^{i+1}_{(\infty)}(X, A)$ if $i \leq n$. One defines cocycles $Z^i_{(\infty)}$, coboundaries $B^i_{(\infty)} = \delta(C^{i-1}_{(\infty)})$, and cohomology groups $H^i_{(\infty)} = Z^i_{(\infty)}/B^i_{(\infty)}$ in the usual way. This definition makes sense for all $i \leq n+1$, since one does not require the finiteness conditions on the $(n+2)$-cells to formulate the condition $\delta h = 0$ for $h \in C^{n+1}_{(\infty)}(X, A)$.

The value of the ℓ_1-homology and ℓ_∞-cohomology groups arises from their quasi-isometry invariance.[2] It is known that the condition that a group G have type F_n is a quasi-isometry invariant [Al][Gr2]. It is also known that if X' and Y' are a $K(G, 1)$ and a $K(G', 1)$, respectively, both with finite $(n+1)$-skeleta, and if the groups G and G' are quasi-isometric, then there are isomorphisms $H^{(1)}_i(X, A) \cong H^{(1)}_i(Y, A)$ for $i \leq n$ and $H^i_{(\infty)}(X, A) \cong H^i_{(\infty)}(Y, A)$ for $i \leq n+1$; here X and Y are the universal covers of X' and Y' respectively. A proof of these facts, constructed along the lines of [Ge2] §11, appears in J. Fletcher's Utah thesis [Fl]. We may thus unambiguously define $H^{(1)}_i(G, A)$ as $H^{(1)}_i(X, A)$ and $H^i_{(\infty)}(G, A)$ as $H^i_{(\infty)}(X, A)$, and we note that the vanishing of either of these groups is a *geometric property* in the sense that it is an invariant of quasi-isometry type.

We shall be interested mainly in the case $A = \mathbb{R}$. For $i \leq n+1$, $Z^{(1)}_i(X, \mathbb{R})$ is a closed subspace of $C^{(1)}_i(X, \mathbb{R})$ since it is the kernel of the bounded linear operator $\hat{\partial} : C^{(1)}_i(X, \mathbb{R}) \to C^{(1)}_{i-1}(X, \mathbb{R})$. However, the image $B^{(1)}_i(X, \mathbb{R})$ need not be a closed subspace. It is usual to define the reduced ℓ_1-homology $\bar{H}^{(1)}_i(X, \mathbb{R})$ to be $Z^{(1)}_i(X, \mathbb{R})/\bar{B}_i(X, \mathbb{R})$, where $\bar{B}_i(X, \mathbb{R})$ is the closure of $B_i(X, \mathbb{R})$ in the normed linear space $C^{(1)}_i(X, \mathbb{R})$ under the ℓ_1-norm, defined in (2.1) above. $\bar{H}^{(1)}_i(X, \mathbb{R})$ is defined for $i \leq n+1$ since the closure operation $\bar{B}_{n+1}(X, \mathbb{R})$ does not depend on the continuity of ∂_{n+2}, and it is quasi-isometry invariant in the range where it is defined.

§3. Distortion.

Let Δ be a subgraph of the graph Γ, and let d_Δ, d_Γ denote their respective "word metrics", so each is a path metric obtained by declaring each edge to be of length 1 (if either graph is disconnected, we allow the value $+\infty$ for two points in different path components). Note that for any pair of points $x, y \in \Delta$ one has $d_\Gamma(x, y) \leq d_\Delta(x, y)$ since every path in Δ is also a path in Γ.

Definition 3.1. We say that the connected subgraph Δ is *undistorted* in Γ (*a.k.a.* the inclusion $\Delta \subset \Gamma$ is a quasi-isometric imbedding) if there exists a constant $K > 0$ (the constant of distortion) so that $d_\Delta(x, y) \leq K\ d_\Gamma(x, y)$ for all $x, y \in \Delta$.

If H is a subgroup of G where both groups are finitely generated, then we can choose a finite set of generators for G containing a set of generators for H as a subset and form the Cayley graphs $\Delta \subset \Gamma$ for H, G, respectively. We say that H is undistorted in G (*a.k.a.* $H < G$ is a quasi-isometric imbedding) if the inclusion $\Delta \subset \Gamma$ is undistorted. Since any two Cayley graphs for the same finitely generated group are bilipschitz equivalent, it follows that the assertion that H is undistorted in G is independent of generators and hence is a group theoretic property.

Remark. In the latter situation one can define the distortion function $f : \mathbb{Z} \to \mathbb{Z}$ for the pair (G, H) by $f(n)$ is the maximum value of $d_\Delta(1, h)$ over all $h \in H$ such that $d_\Gamma(1,h) \leq n$.[3] The subgroup H is undistorted in G iff f is bounded by a linear function. Under change of generators only the equivalence class of f makes sense, where two functions f, g are called equivalent if there are constants $A, B, C, D > 0$ so that $f(n) \leq Ag(Bn)$ and $g(n) \leq Cf(Dn)$ for all $n \in \mathbb{Z}$. Thus one can speak of quadratic, polynomial, exponential, etc., distortion for subgroups.

Examples.

3.1.1. If H is a finitely generated subgroup of the hyperbolic group G, then it is well-known that H is undistorted in H iff H is a quasi-convex subgroup of G, in which case it follows that H is hyperbolic. However, it is possible for a hyperbolic group H to be distorted as a subgroup of the hyperbolic group G. For example, it is known [BF] that if F is a finitely generated free group, then $G = F \,{}_{\mathbb{Z}_\phi}\, \mathbb{Z}$, where $\phi \in \mathrm{Aut}(F)$, is hyperbolic iff G contains no $\mathbb{Z}^2$-subgroup; in this case the word metric on F is exponentially distorted with respect to the word metric on G.

3.1.2. If H is a finitely generated subgroup of the finitely presented group G and if G has a solvable word problem, then the distortion function for the pair (G, H) is recursive iff the Magnus (or generalized word) problem for (G, H) is solvable, [Fa2] Theorem 2.1. A well-known example due to Mihailova shows that there is a finitely generated subgroup H of $F_2 \times F_2$ so that the Magnus problem for the pair is not solvable [L-S] p. 194.

We are interested in deciding whether or not the word metric of a subgraph of a graph is undistorted, or, in the group theoretic situation, when the distortion function is equivalent to a linear function. The following is the main result of this section.

Theorem 3.2. *Let $i : \Delta \subset \Gamma$ be the inclusion of a* connected *subgraph Δ in the graph Γ. Then i is undistorted for their respective word metrics iff the induced homomorphism $H_0^{(1)}(i) : H_0^{(1)}(\Delta, \mathbb{R}) \to H_0^{(1)}(\Gamma, \mathbb{R})$ is injective.*

An immediate consequence is

Corollary 3.3. *If H is a finitely generated subgroup of the finitely generated group G, then H is undistorted in G iff the induced homomorphism $H_0^{(1)}(H, \mathbb{R}) \to H_0^{(1)}(G, \mathbb{R})$ is injective.*

Before giving the proof of Theorem 3.2 we give an application and some examples. Let Δ be the Cayley graph of $\mathbb{Z}$ for the generator 1, so Δ is the nonnegative reals with vertices at the integer points. If Γ is a graph with vertex set $V(\Gamma)$, a *pseudo-ray* is a lipschitz function $f : \mathbb{Z} \to V(\Gamma)$. It is easily seen that this is the condition for the induced map $H_0^{(1)}(\mathbb{Z}, \mathbb{R}) \to H_0^{(1)}(\Gamma, \mathbb{R})$ to exist. The pseudo-ray $f : \mathbb{Z} \to \Gamma$ is called a quasi-geodesic ray if it is a quasi-isometric imbedding. Recall that this means that there are constants $\lambda \geq 1,\ \epsilon \geq 0$ so that

$$\frac{1}{\lambda}|n-m| - \epsilon \leq d_\Gamma(f(n), f(m)) \leq \lambda|n-m| + \epsilon.$$

[3] I called this the "rewrite function" in [Ge6], whereas Gromov called $f(n)/n$ the distortion in [Gr2]. Gromov's term distortion has now become standard, but it has become customary since Farb's paper [Fa2] to omit the factor of n in the denominator, so my rewrite function is now called the distortion function.

Corollary 3.4. *The pseudo-ray $f : \mathbb{Z} \to \Gamma$ is a quasi-geodesic ray iff the induced map $H_0^{(1)}(f)$ is injective.*

Proof. For a graph Γ and natural number p we define the graph $\Gamma(p)$ to be obtained from Γ by attaching a new edge $e(v, w)$ from vertex v to vertex w for each pair of vertices v, w with $d_\Gamma(v, w) \leq p$. It is readily checked that the inclusion $\Gamma \subset \Gamma(p)$ is undistorted and is in fact a quasi-isometry. It follows that the induced map on $H_0^{(1)}$ is an isomorphism.

Now let $f : \mathbb{Z} \to V(\Gamma)$ be a pseudo-ray. It follows that there is a number $p \geq 1$ so that $d_\Gamma(f(n), f(n+1)) \leq p$ for all numbers n. Thus there is an edge of $\Gamma(p)$ joining vertices $f(n)$ and $f(n+1)$, so the map $f : \mathbb{Z} \to V(\Gamma)$ extends to a map $f(p) : \Delta \to \Gamma(p)$, where Δ is the Cayley graph of $\mathbb{Z}$.

Suppose now that f is a quasi-geodesic ray. Then one sees that there is a number $q > 0$ so that if $|n - m| \geq q$ then $f(n) \neq f(m)$. In particular, the restriction of f to $q\mathbb{Z}$ is injective. Thus the composition $q\mathbb{Z} \subset \mathbb{Z} \to V(\Gamma) = V(\Gamma(pq))$ extends to an *injective* map F of graphs $\Delta \to \Gamma(pq)$ which is undistorted for their word metrics and hence induces an injection on $H_0^{(1)}$ by the theorem. Since $q\mathbb{Z} \subset \mathbb{Z}$ and $\Gamma \subset \Gamma(pq)$ are quasi-isometries and hence induce isomorphisms on $H_0^{(1)}$, it follows that the original map f induces an injection on $H_0^{(1)}$.

For the converse, assume that the pseudo-ray $f : \mathbb{Z} \to V(\Gamma)$ induces an injection on $H_0^{(1)}$. Let p be chosen so that f extends to a map $f(p) : \Delta \to \Gamma(p)$; it follows that $f(p)$ also induces an injection on $H_0^{(1)}$. We claim that there is a number $q > 0$ so that the restriction of $f(p)$ to $q\mathbb{Z}$ is injective. If this is the case, then we get an *injective* extension F of $f(p)|_{q\mathbb{Z}}$ to a map $\Delta \to \Gamma(pq)$; the theorem shows that F does not distort the word metrics, and it follows that f does not distort word metrics.

So it remains to establish the claim, that there exists $q > 0$ so that $f(p)|_{q\mathbb{Z}}$ is injective. We assume the contrary and derive a contradiction. It will be convenient to denote elements of $\mathbb{Z}$ by monomials x^n, $n \in \mathbb{Z}$, so as not to confuse them with coefficients in chains. We have then in particular that for each $n > 0$ there exists $k_n > n^2$ and $m_n > 0$ so that $f(m_n) = f(m_n + k_n)$.

Consider the 0-chain $\zeta = \sum_{n>0} (x^{m_n + k_n} - x^{m_n})/n^2$ which is clearly summable, so ζ represents a class in $H_0^{(1)}(\mathbb{Z}, \mathbb{R}) = H_0^{(1)}(\Delta, \mathbb{R})$. But $\zeta \notin B_0^{(1)}(\Delta, \mathbb{R})$, since $Z_1^{(1)}(\Delta, \mathbb{R}) = 0$ as Δ is a tree and since the only filling of ζ is not summable, as follows from $\sum_{n>0} k_n/n^2 = \infty$. Thus the class of ζ in $H_0^{(1)}(\Delta, \mathbb{R})$ is nonzero. On the other hand ζ clearly maps to zero under f, and we deduce that the map induced by f on $H_0^{(1)}$ is not injective, giving the desired contradiction. This establishes the claim, and the proof of the corollary is complete.

Example. We give the calculation of $H_0^{(1)}(\mathbb{Z}, \mathbb{R})$. Let α be a summable sequence of real numbers, so $\alpha = (a_0, a_1, a_2, \dots)$ where $\sum_n |a_n| < \infty$. We define the first difference $\Delta\alpha = (a_0, a_1 - a_0, a_2 - a_1, \dots)$. Clearly $\Delta\alpha$ is summable if α is summable.

Proposition 3.5. *$H_0^{(1)}(\mathbb{Z}, \mathbb{R})$ is the quotient space of the vector space of summable sequences modulo the subspace of first differences of summable sequences.*

This follows from the observation that $C_0^{(1)}(\mathbb{Z}, \mathbb{R})$ consists of summable sequences whereas $B_0^{(1)}(\mathbb{Z}, \mathbb{R})$ consists of first differences of summable sequences.

Let $0 < p \leq 1$ and let $\alpha_p = \Delta\{n \mapsto 1/n^p\}$. Then one can check that the classes of α_p are linearly independent in $H_0^{(1)}(\mathbb{Z}, \mathbb{R})$.

Here are two examples to show how Corollary 3.3 detects distortion in groups.

Example. Let $G = \langle x, y \mid yxy^{-1} = x^2 \rangle$, $H = \langle x \rangle$, so Δ is the Cayley graph of H and Γ is that of G for given generators. Then $d_\Gamma(1, x^{2^n}) = 2n + 1$, so the distortion function is exponential. To see what is happening homologically, let $\zeta = \sum_{n=1}^\infty (x^{2^n} - 1)/n^3 \in C_0^{(1)}(\Delta)$. Then $\zeta \notin B_0^{(1)}(\Delta)$ since $\sum 2^n/n^3 = \infty$, but $\zeta \in B_0^{(1)}(\Gamma)$ since $\sum (2n+1)/n^3 < \infty$. Thus the class of ζ is a nonzero element of the kernel of the map $H_0^{(1)}(\Delta) \to H_0^{(1)}(\Gamma)$.

We know more about this example than just that the inclusion $\Delta \subset \Gamma$ is distorted: the group theory tells us that the map is exponentially distorted. The ℓ_1-homology as defined is incapable to detecting the exponential nature of the distortion. However, if one considers modifications of the theory which allow for growth of cycles, then it is possible to obtain this finer information. That is the topic we hope to address in a future paper.

Example. Let $G = \langle x, y, z \mid [z, x],\ [z, y],\ z = [x, y] \rangle$, $H = \langle z \rangle$, so Δ is the Cayley graph of H and Γ is the Cayley graph of G. One has $d_\Gamma(1, z^{n^2}) = 4n$ so H is quadratically distorted in G. The distortion can be detected homologically by setting $\zeta = \sum_{n=1}^\infty (z^{n^2} - 1)/n^3 \in C_0^{(1)}(\Delta)$. Then $\zeta \notin B_0^{(1)}(\Delta)$ since $\sum n^2/n^3 = \infty$, but $\zeta \in B_0^{(1)}(\Gamma)$ since $\sum 4n/n^3 < \infty$. Thus the class of ζ is a nonzero element of the kernel of the map $H_0^{(1)}(\Delta) \to H_0^{(1)}(\Gamma)$.

As in the preceding example, it requires finer invariants to detect the quadratic nature of the distortion homologically.

We proceed now to the proof of Theorem 3.2. The argument depends on the approximation theorem of [AG].

Theorem [AG]**.** *If Γ is a graph then every summable 1-cycle $z \in Z_1^{(1)}(\Gamma, \mathbb{R})$ can be written as a* coherent *sum*

$$(*) \qquad z = \sum_{i=1}^\infty a_i z_i,$$

where $a_i > 0$ and where z_i are simple edge circuits of Γ.

The coherence condition means that for all oriented edges e one has $z_i(e) \cdot z_j(e) \geq 0$ for all i, j, where $z_i(e)$ is the coefficient of e in z_i; thus there are no cancellations in any of the sums involved in $(*)$. It follows that the convergence in $(*)$ is monotone, so $|z|_1 = \sum_i a_i |z_i|_1 = \sum_i a_i \text{ length}(z_i)$.

First assume that the inclusion $i : \Delta \to \Gamma$ of graphs is undistorted for their word metrics; we must prove that $H_0^{(1)}(i)$ is injective. So suppose $\alpha \in C_0^{(1)}(\Delta)$ is such that there exists $\beta \in C_1^{(1)}(\Gamma)$ with $\alpha = \hat{\partial}\beta$ in $C_0^{(1)}(\Gamma)$. We must prove that there exists $\beta_1 \in C_1^{(1)}(\Delta)$ with $\hat{\partial}\beta_1 = \alpha$. We need

Lemma 3.6. *Suppose that $\beta \in C_1^{(1)}(\Gamma)$ is such that $\hat{\partial}\beta \in C_0^{(1)}(\Delta)$. Then there exists a coherent family of simple closed edge circuits z_j and simple arcs A_i of Γ with $\partial A_i \in V(\Delta)$ so that*

$$\beta = \sum y_i A_i + \sum y_j z_j$$

where $u_i, v_j > 0$.

Assume the lemma for the moment, and we finish the proof that $H_0^{(1)}(i)$ is injective. We calculate $\alpha = \hat{\partial}\beta = \hat{\partial}(\sum u_i A_i + \sum v_j z_j) = \sum u_i \hat{\partial} A_i$. Since the word metrics are undistorted, there exists $K > 0$ and there exist edge-paths A_i' in Δ connecting the end points of A_i so that $\text{length}(A_i') \leq K\ \text{length}(A_i)$; note that the connectivity of Δ is used at this point to assure that the end points of A_i can indeed be be connected in Δ. Let $\beta_1 = \sum u_i A_i'$ and calculate $|\beta_1|_1 \leq \sum u_i\ \text{length}(A_i') \leq K \sum u_i\ \text{length}(A_i) \leq K|\beta|_1 < \infty$. Hence $\beta_1 \in C_1^{(1)}(\Delta)$. Next calculate $\alpha = \sum u_i\ \hat{\partial} A_i = \sum u_i\ \hat{\partial} A_i' = \hat{\partial}\beta_1$, as desired.

Proof of Lemma 3.6. We have $\beta \in C_1^{(1)}(\Gamma)$ with $\hat{\partial}\beta \in C_0^{(1)}(\Delta)$. Let $\bar{\Gamma}$ be the graph obtained from Γ by collapsing all vertices of Δ to a single vertex and let $p : \Gamma \to \bar{\Gamma}$ be the projection. Denote by p_* the induced map on chains.

Note that $\hat{\partial} p_* \beta = p_* \hat{\partial} \beta = p_* \alpha$. We claim that $p_* \alpha = 0$, so that $p_* \beta$ is a summable 1-cycle on $\bar{\Gamma}$. To see this consider the commutative diagram

$$\begin{array}{ccccc}
 & & C_1^{(1)}(\Gamma) & \xrightarrow{p_*} & C_1^{(1)}(\bar{\Gamma}) \\
 & & \downarrow \hat{\partial} & & \downarrow \hat{\partial} \\
C_0^{(1)}(\Delta) & \longrightarrow & C_0^{(1)}(\Gamma) & \xrightarrow{p_*} & C_0^{(1)}(\bar{\Gamma}) \\
 & & & & \downarrow \pi \\
 & & & & C_0^{(1)}(\text{point}) \cong \mathbb{R}
\end{array}$$

where the map $C_0^{(1)}(\bar{\Gamma}) \xrightarrow{\pi} C_0^{(1)}(\text{point})$ is obtained by collapsing all vertices to a single point. The element $\hat{\partial}\beta$ of $C_0^{(1)}(\Gamma)$ is a sum $\sum a_i v_i$ where $v_i \in V(\Gamma)$ and $\sum |a_i| < \infty$; since $\hat{\partial}\beta = \alpha$, it follows that the only coefficients $a_i \neq 0$ are such that $v_i \in V(\Delta)$. Thus $\sum a_i$ is the only potentially nonzero coefficient of a point in $p_* \hat{\partial} \beta \in C_0^{(1)}(\bar{\Gamma})$. However $\sum a_i = 0$ since the composition $\pi \circ \hat{\partial} : C_1^{(1)}(\Gamma) \to C_0^{(1)}(\text{point})$ is zero. Thus $p_* \hat{\partial} \beta = 0$ and the claim is established.

It follows from the approximation theorem of [AG] quoted above that $p_* \beta = \sum u_i \zeta_i$, where ζ_i is a simple edge-circuit of $\bar{\Gamma}$, $u_i > 0$, and where the family is coherent. Now pull back the circuit ζ_i to Γ. We either get a simple arc A_i with boundary points in Δ if ζ_i meets $p(\Delta)$ or we get a simple edge-circuit z_j of Γ if ζ_j does not meet $p(\Delta)$. Since p is injective on edges, we get $\beta = \sum u_i A_i + \sum v_j z_j$ with $u_i, v_j > 0$, as desired. This completes the proof of the lemma.

Before proceeding to the converse we need another consequence of the approximation theorem. Let X be an arbitrary graph, and consider $B_0(X, \mathbb{R}) = B_0$ as a normed linear space equipped with the filling norm[4] induced from the boundary map $\partial : C_1 \to B_0$. Let $\hat{B}_0$ be the completion of B_0 for the filling norm, and recall that $B_0^{(1)}$ is the image of the completed map $\hat{\partial} : C_1^{(1)} \to C_0^{(1)}$. There is a canonical map $\hat{B}_0 \to B_0^{(1)}$ arising from properties of completion.

[4]If X is a graph, the boundary map $\partial : C_1(X, \mathbb{Z}) \to C_0(X, \mathbb{Z})$ is a bounded linear map with norm 2. If $b \in B_0(X, \mathbb{Z}) = \text{Image}(\partial)$, we define the *filling norm* $N\mathbb{R}(b) = \inf\{|c|_1 \mid c \in C_1(X, \mathbb{Z}),\ \partial c = b\}$. It is easy to see that $N\mathbb{R}$ enjoys all the usual properties of a norm on a real

Lemma 3.7. *The canonical map $\hat{B}_0 \to B_0^{(1)}$ is an isomorphism of normed linear spaces. In particular, $B_0^{(1)}$ is complete, so it is a Banach space.*

Proof. We have $B_0 = C_1/Z_1$ with Z_1 closed in C_1. It follows that $\hat{B}_0 = (C_1/Z_1)\hat{} = C_1^{(1)}/\hat{Z}_1$. Also we have $B_0^{(1)} = C_1^{(1)}/Z_1^{(1)}$. But by the approximation theorem $Z_1^{(1)} = \hat{Z}_1$, whence $\hat{B}_0 = B_0^{(1)}$, and the lemma is established.

Now we can proceed to the converse direction in Theorem 3.2, so we suppose that $H_0^{(1)}(i)$ is injective. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_0^{(1)}(\Delta) & \longrightarrow & C_0^{(1)}(\Delta) & \longrightarrow & H_0^{(1)}(\Delta) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & B_0^{(1)}(\Gamma) & \longrightarrow & C_0^{(1)}(\Gamma) & \longrightarrow & H_0^{(1)}(\Gamma) & \longrightarrow & 0 \end{array}$$

Here α, β, γ are induced by the inclusion $i : \Delta \subset \Gamma$. Note that α and β are injective for obvious reasons whereas $\gamma = H_0^{(1)}(i)$ is injective by hypothesis.

From the snake lemma it follows that the induced map coker $\alpha \to$ coker β is injective, or, equivalently, if $x \in B_0^{(1)}(\Gamma)$ and x maps to an element of the image of β, then x is in the image of α. But this means that the canonical map $p : B_0^{(1)}(\Delta) \to B_0^{(1)}(\Gamma) \times_{C_0^{(1)}(\Gamma)} C_0^{(1)}(\Delta) = P$ is 1-1 and onto, where P is the fibre product.

Now both $B_0^{(1)}(\Delta)$ and P are Banach spaces and the map p above is continuous. For $B_0^{(1)}(\Delta)$ (and similarly $B_0^{(1)}(\Gamma)$) this follows from Lemma 3.7. That P is a Banach space follows from the exact sequence

$$0 \to P \to B_0^{(1)}(\Gamma) \oplus C_0^{(1)}(\Delta) \to C_0^{(1)}(\Gamma),$$

so P is a closed subspace of a Banach space.

Thus we have a continuous map $p : B_0^{(1)}(\Delta) \to P$ between Banach spaces which is 1-1 and onto. It follows from the Banach inversion theorem that the inverse of p is continuous, and hence p is an isomorphism in the category of Banach spaces and continuous linear maps.

Since $\beta : C_0^{(1)}(\Delta) \to C_0^{(1)}(\Gamma)$ is an isometric imbedding, it follows that the composition $B_0^{(1)}(\Delta) \cong P \to B_0^{(1)}(\Gamma)$ is *undistorted*, where the second arrow is the projection of $P = B_0^{(1)}(\Gamma) \times_{C_0^{(1)}(\Gamma)} C_0^{(1)}(\Delta)$ on the first factor. Here we recall from [Ge2] that a continuous injective homomorphism $f : V \to W$ of normed linear spaces over $\mathbb{R}$ is called undistorted if the norm of W restricted to V (via the injection f) is equivalent to the norm on V. Hence $i_* : B_0^{(1)}(\Delta) \to B_0^{(1)}(\Gamma)$ is undistorted.[5]

But if v, w are vertices of Δ, then the filling norm of $v - w$ in $B_0^{(1)}(\Delta)$ is equal to $d_\Delta(v, w)$, and for the same reason the filling norm in $B_0^{(1)}(\Gamma)$ is $d_\Gamma(v, w)$. This actually involves a theorem, which says that for integral boundaries in dimension

[5]More generally, the homomorphism $f : A \to B$ of normed abelian groups is said to be

zero the integral filling norm is equal to the real filling norm, [Ge4] Theorem 9.1. It follows that there exists $K > 0$ so that

$$\frac{1}{K} d_\Delta(v, w) \le d_\Gamma(v, w) \le K d_\Delta(v, w)$$

for all $v, w \in V(\Delta)$. It follows that the inclusion $\Delta \subset \Gamma$ is undistorted, and the proof of Theorem 3.2 is complete.

For the record, we make explicit the remarks above about filling norms in dimension zero in the following

Corollary 3.8. *The following are equivalent for the connected subgraph Δ of the graph Γ.*

(1) *The inclusion $i : \Delta \subset \Gamma$ is a quasi-isometric imbedding.*
(2) $H_0^{(1)}(i) : H_0^{(1)}(\Delta, \mathbb{R}) \to H_0^{(1)}(\Gamma, \mathbb{R})$ *is injective.*
(3) $i_* : B_0^{(1)}(\Delta, \mathbb{R}) \to B_0^{(1)}(\Gamma, \mathbb{R})$ *is undistorted as a map of Banach spaces.*
(4) $i_* : B_0(\Delta, \mathbb{R}) \to B_0(\Gamma, \mathbb{R})$ *is undistorted as a map of normed linear spaces for their respective filling norms.*
(5) $i_* : B_0(\Delta, \mathbb{Z}) \to B_0(\Gamma, \mathbb{Z})$ *is undistorted as a map of normed abelian groups for their respective filling norms.*

Remark. It follows from the preceding corollary that to check (5) it suffices to check it on special elements of $B_0(\Delta, \mathbb{Z})$ of the form $v - w$, where $v, w \in V(\Delta)$. The statement that (1) and (5) are equivalent is a consequence of Theorem 3.2 which can be understood without mention of homology, although its proof involves the homological machinery.

Remark. If Δ is not connected in Corollary 3.8, then we *define* the subgraph Δ of Γ to be undistorted if the induced map $B_0(\Delta, \mathbb{Z}) \to B_0(\Gamma, \mathbb{Z})$ is undistorted as a homomorphism of normed abelian groups, each equipped with its respective filling norm. This agrees with the earlier notion if Δ is connected by the preceding corollary.

If Δ is not connected then the converse direction of the proof of Theorem 3.2 remains valid without change. As a consequence we obtain

Corollary 3.9. *If $i : \Delta \to \Gamma$ is the inclusion of a (not necessarily connected) subgraph and if the induced homomorphism $H_0^{(1)}(i) : H_0^{(1)}(\Delta) \to H_0^{(1)}(\Gamma)$ is injective, then the induced map $B_0(\Delta, \mathbb{Z}) \to B_0(\Gamma, \mathbb{Z})$ is undistorted as a homomorphism of normed abelian groups, and consequently $i : \Delta \to \Gamma$ is undistorted.* $\mathbb{Z}$

Remark. The true situation in case Δ is disconnected will be discussed in the next section; this requires introduction of a new homology functor.

§4. Distortion for general subgraphs.

Definition 4.1. If Δ is a subgraph of the graph Γ we say that Δ is undistorted in Γ if the induced homomorphism of normed abelian groups $B_0(\Delta, \mathbb{Z}) \to B_0(\Gamma, \mathbb{Z})$ is undistorted for their respectective filling norms. By Corollary 3.8 this agrees with the usual definition in terms of word metrics if Δ is connected.

Example 4.1.1. Disconnected subgraphs of interest arise as follows. Let H be a finitely generated subgroup of the finitely generated group G, and let A be a finite set of generators for G containing the subset B of generators for H. Let $\Gamma = \Gamma_{G,A}$ be the Cayley graph of G and let $\Gamma_{H,B}$ be the Cayley graph of H, viewed as a subgraph of Γ containing the base point. Let $\Delta = \cup_{g\in G}\, g\Gamma_{H,B}$, so Δ is the union of all translates of $\Gamma_{H,B}$ under the left action of G. B. Farb [Fa1] has defined the pair (G, H) to be relatively hyperbolic if the quotient graph $\Gamma//\Delta$ is hyperbolic with its path metric assigning all edges length 1, where $\Gamma//\Delta$ collapses each connected component of Δ to its own vertex. We shall return to this example in Theorem 5.9 below.

Recall from §2 that $\bar{H}_0^{(1)}(\Delta, \mathbb{R}) = C_0^{(1)}/\bar{B}_0^{(1)}$. If S is a discrete set let $\ell_1(S)$ denote the set of ℓ_1-summable real valued functions on S. Such a function can be viewed as a formal sum $\sum_{s\in S} a_s s$, where $\sum_s |a_s| < \infty$.

Lemma 4.2. *We have* $\bar{H}_0^{(1)}(\Delta, \mathbb{R}) \cong \ell_1(\pi_0(\Delta))$.

Proof. We have a map $\eta : C_0^{(1)}(\Delta, \mathbb{R}) \to \ell_1(\pi_0(\Delta))$ given by $\sum_{v\in V(\Delta)} a_v v \mapsto \sum_{v\in V(\Delta)} a_v [v]$, where $[v]$ denotes the path component of the vertex v. The map η is clearly a surjective homomorphism whose kernel is the set of summable functions $\sum a_v v$ such that $\sum_{x\in[v]} a_x = 0$ for all $[v] \in \pi_0(\Delta)$.

If Δ_v denotes the connected component of the vertex v in Δ, it suffices to prove that every sum $\alpha = \sum a_x x$, where $x \in [v]$, $\sum_x |a_x| < \infty$, $\sum_x a_x = 0$, can be approximated in ℓ_1-norm by an element of $B_0^{(1)}(\Delta_v, \mathbb{R})$. To this end choose a base point $x_0 \in [v]$ and let $\epsilon > 0$. Choose $S = \{x_0, x_1, x_2, \ldots, x_n\} \subset [v]$ such that $\sum_{x\notin S} |a_x| < \epsilon$, and let $\beta = -(\sum_{i=1}^n a_{x_i})x_0 + \sum_{i=1}^n a_{x_i} x_i$. Then $\beta \in B_0(\Delta_v, \mathbb{R})$ and $|\alpha - \beta|_1 < 2\epsilon$. This completes the proof.

Definition 4.3. We define $I_0(\Delta)$ to be the kernel of the projection $H_0^{(1)}(\Delta, \mathbb{R}) \to \bar{H}_0^{(1)}(\Delta, \mathbb{R})$. The proof of the lemma shows that $I_0(\Delta)$ consists of cosets modulo $B_0^{(1)}(\Delta, \mathbb{R})$ of sums $\alpha = \sum_{v\in V(\Delta)} a_v v$ such that $\sum_{x\in[v]} a_x = 0$ for all vertices v of Δ. We have the short exact sequence

$$0 \to I_0(\Delta) \to H_0^{(1)}(\Delta, \mathbb{R}) \to \bar{H}_0^{(1)}(\Delta, \mathbb{R}) \to 0. \tag{4.3.1}$$

The term $\bar{H}_0^{(1)}(\Delta, \mathbb{R}) \cong \ell_1(\pi_0(\Delta))$ contains topological information about connected components while $I_0(\Delta)$ contains metric information about the word metric, as we shall see. Note also the short exact sequence

$$0 \to B_0^{(1)}(\Delta, \mathbb{R}) \to C_0'(\Delta) \to I_0(\Delta) \to 0, \tag{4.3.2}$$

where $C_0'(\Delta)$ is defined to be the set of sums $\sum a_v v \in C_0^{(1)}(\Delta)$ such that $\sum_{x\in[v]} a_x = 0$ for all $v \in V(\Delta)$. It is a closed subspace of $C_0^{(1)}(\Delta)$ for the ℓ_1-norm. Note however that $B_0^{(1)}(\Delta, \mathbb{R})$ is not a closed subset in general and in fact its closure is all of $C_0'(\Delta, \mathbb{R})$, again by the lemma.

I_0 is a functor from graphs to real vector spaces.

Example. If Δ is connected then $\bar{H}_0^{(1)}(\Delta, \mathbb{R}) = \mathbb{R}$, so $I_0(\Delta)$ is the kernel of the canonical homomorphism $H_0^{(1)}(\Delta, \mathbb{R}) \to \mathbb{R}$.

We can now state the main result of this section.

Theorem 4.4. *If $i : \Delta \to \Gamma$ is the inclusion of a subgraph Δ in the graph Γ, then i is undistorted iff the induced homomorphism $I_0(i) : I_0(\Delta) \to I_0(\Gamma)$ is injective.*

Before beginning the proof, we give two examples.

Examples.

4.4.1. Let Γ be the unit square lattice in the plane so Γ is the Cayley graph of the presentation $\mathcal{P} = \langle x, y \mid [x, y]\rangle$. Let Δ consist of the union of the portion of the x-axis to the right of $(1, 0)$ with the portion of the y-axis above $(0, 1)$. Thus there are two connected components to Δ. One can check that the inclusion $\Delta \subset \Gamma$ is undistorted, and hence the induced map on I_0 is injective. On the other hand the induced map on $\bar{H}_0^{(1)}$ is not injective since Γ has only one connected component.

4.4.2. As a second example, take the same Γ and let Δ be the union of the portion of the x-axis to the right of $(1, 0)$ with the portion of the line $y = 1$ to the right of the point $(1, 1)$. In this case the inclusion $\Delta \subset \Gamma$ is distorted. To see this, let $\alpha_n \in B_0(\Delta, \mathbb{Z})$ be given by $\alpha_n = (n, 0) - (1, 0) + (1, 1) - (n, 1)$. Then $|\alpha_n|_{\Delta,\text{fill}} = 2(n-1)$ for all $n > 0$, whereas $|\alpha_n|_{\Gamma,\text{fill}} = 2$ for all $n \geq 2$.

Proof of Theorem 4.4. If $I_0(i)$ is injective, then the arguments for the converse direction of Theorem 3.2 apply by replacing $H_0^{(1)}$ by I_0 and $C_0^{(1)}$ by C_0'. The result is that the map $B_0(\Delta, \mathbb{Z}) \to B_0(\Gamma, \mathbb{Z})$ is undistorted.

Conversely, assume that the homomorphism $B_0(\Delta, \mathbb{Z}) \to B_0(\Gamma, \mathbb{Z})$ is undistorted and let $\alpha \in C_0'(\Delta)$ be such that its image in $C_0'(\Gamma)$ lies in the subspace $B_0^{(1)}(\Gamma, \mathbb{R})$. We want to show that $\alpha \in B_0^{(1)}(\Delta, \mathbb{R})$.

Let $\alpha = \hat{\partial}\beta$, $\beta \in C_1^{(1)}(\Gamma, \mathbb{R})$. Let $p : \Gamma \to \bar{\Gamma}$ collapse $V(\Delta_v)$ to one point for each vertex v, where Δ_v is the connected component of v. Thus if $\Delta_v \neq \Delta_{v'}$, then $p(V(\Delta_v)) \neq p(V(\Delta_{v'}))$. Note that the map p is bijective on edges, so any 1-chain on $\bar{\Gamma}$ lifts uniquely to a 1-chain on Γ. We need

Lemma 4.5. $p_*\beta \in Z_1^{(1)}(\bar{\Gamma}, \mathbb{R})$.

Proof. Since $\hat{\partial}\beta$ is supported on Δ, the only potential vertices with nonzero coefficients in $p_*\hat{\partial}\beta$ are of the form $p(V(\Delta_v))$. But this coefficient is also zero since $\hat{\partial}\beta = \alpha \in C_0'(\Delta)$, proving the lemma.

Thus we can apply the approximation theorem of [AG] to $p_*\beta$ to write it in the form $\sum_j u_j z_j$, a coherent sum of simple circuits z_j on $\bar{\Gamma}$, $u_j > 0$. Pull back z_j to Γ to obtain a collection of simple arcs and simple circuits, and let ζ_j be the integral 1-chain on Γ determined by them; note that we can ignore any simple circuits in this collection since their boundaries are zero, so we assume from now on that there are only simple arcs occurring.

Lemma 4.6. $\partial\zeta_j \in B_0(\Delta, \mathbb{Z})$.

Proof. Let $[v_1], [v_2], \ldots, [v_n]$ be the vertices of $\bar{\Gamma}$ in order where z_j meets $p(V(\Delta))$, and let $\bar{A}_{ji}$ be the arc joining $[v_i]$ to $[v_{i+1}]$ on z_j, where the indices i are taken modulo n. All this makes sense since z_j is a simple circuit on $\bar{\Gamma}$. Then $\bar{A}_{ji}$ lifts to a simple arc A_{ji} on Γ with end points in Δ. Note that the terminal point $\tau(A_{ji})$ of A_{ji} lies in the same connected component of Δ as the initial point $\iota(A_{j,i+1})$, with indices i taken modulo n. Since $\zeta_j = \sum_i A_{ji}$ we have

$$\begin{aligned}\partial\zeta_j &= \sum_{i \pmod n} (\tau(A_{ji}) - \iota(A_{ji})) \\ &= \sum_{i \pmod n} (\tau(A_{ji}) - \iota(A_{j,i+1})) \in B_0(\Delta, \mathbb{Z}),\end{aligned}$$

proving the lemma.

Since the map $B_0(\Delta,\mathbb{Z}) \to B_0(\Gamma,\mathbb{Z})$ is undistorted there exists $K > 0$ so that for all $\gamma \in B_0(\Delta,\mathbb{Z})$ one has $|\gamma|_{\Delta,\text{fill}} \le K|\gamma|_{\Gamma,\text{fill}}$, where $|\gamma|_{\Delta,\text{fill}}$ denotes the filling norm in Δ. It follows that there exists $\zeta'_j \in C_1(\Delta,\mathbb{Z})$ so that $\partial\zeta'_j = \partial\zeta_j$ and such that $|\zeta'_j|_1 \le K|\zeta_j|_1$. Define $\beta' = \sum_j u_j\zeta'_j$ and calculate $|\beta'|_1 \le \sum_j u_j|\zeta'_j|_1 \le K\sum_j u_j|\zeta_j|_1 = K|\beta|_1 < \infty$, whence $\beta' \in C_1^{(1)}(\Delta,\mathbb{R})$. Since $\partial\beta' = \partial\beta = \alpha$, it follows that $\alpha \in B_0^{(1)}(\Delta,\mathbb{R})$, and the proof of the theorem is complete.

Remark. Theorem 3.2 is a consequence of Theorem 4.4 since if Δ is connected and $\alpha \in C_0^{(1)}(\Delta,\mathbb{R}) \cap B_0^{(1)}(\Gamma,\mathbb{R})$, then it follows that $\alpha \in C'_0(\Delta,\mathbb{R})$ since the sum of the coefficients of a 0-boundary is zero. Then Theorem 4.4 says $\alpha \in B_0^{(1)}(\Delta,\mathbb{R})$, which is the conclusion of Theorem 3.2.

The fact that for a connected graph Δ the inclusion $\Delta \subset \Gamma$ does not distort the word metric iff the inclusion $B_0(\Delta,\mathbb{Z}) \subset B_0(\Gamma,\mathbb{Z})$ is undistorted can be viewed as a consequence of the fact that the number n in the proof of Lemma 4.6 is at most equal to 1. Thus to detect whether general 0-boundaries are distorted, it suffices to check whether the distances between two vertices in Δ are distorted in Γ.

This last fact admits a generalization in the following result, which serves to give a more geometric criterion to determine when a general subgraph Δ of a graph Γ is undistorted.

Theorem 4.7. *A necessary and sufficient condition for a (disconnected) subgraph Δ of the graph Γ to be undistorted is that there exist $K > 0$ so that for each finite collection of oriented geodesic segments γ_i, $1 \le i \le n$, lying in distinct connected components Δ_i of Δ, one has*

$$\sum_{i=1}^n \text{length}(\gamma_i) \le K\ |\sum_{i=1}^n \partial\gamma_i|_{\Gamma,\text{fill}}. \tag{4.7.1}$$

In other words, up to the uniform multiplicative constant K the 0-boundary $\sum_{i=1}^n \partial\gamma_i$ fills as efficiently in Δ as it does in Γ. The point of this result is the restricted nature of the 0-boundaries supported in the various connected components of Δ which occur in (4.7.1).

Proof of Theorem 4.7. The proof amounts to a commentary on the proof of Theorem 4.4, in making more precise exactly what was established there. One direction is clear, since if the inclusion $\Delta \subset \Gamma$ is undistorted, then (4.7.1) holds. So we assume that (4.7.1) holds and proceed to prove that the induced homomorphism $I_0(\Delta) \to I_0(\Gamma)$ is injective.

Let $\alpha \in C'_0(\Delta,\mathbb{R})$ be such that $\alpha = \hat\partial\beta$ with $\beta \in C_1^{(1)}(\Gamma,\mathbb{R})$. As in the proof of Theorem 4.4 let $\pi : \Gamma \to \bar\Gamma$ collapse the vertex set of each connected component of

Δ to its own point, so $p_*\beta \in Z_1^{(1)}(\Gamma,\mathbb{R})$. The approximation theorem then allows us to write $p_*\beta$ as a coherent sum $\sum_j u_j z_j$, where $u_j > 0$ and where z_j are simple circuits on $\bar\Gamma$. Consider one of these circuits z_j, and let $[v_1],[v_2],\ldots,[v_n]$ be the vertices of $\bar\Gamma$ in order where z_j meets $p(V(\Delta))$. Note that none of these vertices is repeated in the circuit since z_j is simple. Let $\bar A_{ji}$ be the arc on z_j joining $[v_i]$ to $[v_{i+1}]$, where the indices i are taken modulo n. Then $\bar A_{ji}$ lifts to a simple arc A_{ji} on Γ, and the terminal vertex $\tau(A_{ji})$ and the initial vertex $\iota(A_{j,i+1})$ lie in the same connected component Δ_i of Δ. Let γ_i be an oriented geodesic segment in Δ_i joining the second vertex to the first. Then by (4.7.1) we have $\sum_{i=1}^n \text{length}(\gamma_i) \le K\sum_{i=1}^n \text{length}(A_{ji}) = K\ |z_j|_1$. If we let $\zeta_j' = \sum_{i=1}^n \gamma_i$, then $\zeta_j' \in B_0(\Delta,\mathbb{Z})$, and $\beta' := \sum_j u_j\zeta_j' \in C_1^{(1)}(\Delta,\mathbb{R})$ with $\hat\partial\beta' = \alpha$. Hence that $\alpha \in B_0^{(1)}(\Delta,\mathbb{R})$, and the induced homomorphism $I_0(\Delta) \to I_0(\Gamma)$ is injective. It follows from Theorem 4.4 that the inclusion $\Delta \subset \Gamma$ is undistorted, and the proof is complete.

Definition 4.8. We call an element $b \in B_0(\Delta,\mathbb{Z})$ *unramified* if for each connected component Δ_0 of Δ one has $|b_{\Delta_0}|_1 \le 2$, where b_{Δ_0} is the part of b supported in Δ_0. Thus b is unramified if its restriction to each connected component of Δ is filled by a geodesic segment in that connected component. In this terminology Theorem 4.7 states that the subgraph Δ of the graph Γ is undistorted iff the unramified integral boundaries in Δ are uniformly undistorted in Γ for their respective filling norms.

§5. Mayer-Vietoris exact sequence and distortion.

Definition 5.1. We say that a CW-complex X has bounded geometry in dimensions up to $n+1$ if there is a constant $M > 0$ so that for each i-cell $e^{(i)}$, $i \le n+1$, the ℓ_1-norm of the $(i-1)$-chain $\partial_i e^{(i)}$ is at most M; here ℓ_1-norms are calculated with respect to a basis of cells in the appropriate dimension. For such a complex X the ℓ_1-homology $H_i^{(1)}(X,\mathbb{R})$ is defined for $i \le n$ by completing the chain groups in the ℓ_1-norms and taking cycles modulo boundaries in the usual way. The reduced ℓ_1-homology groups $\bar H_i^{(1)}(X,\mathbb{R})$ are also defined for $i \le n+1$ in the usual way.

Examples. Every simplicial complex has bounded geometry in all dimensions. If X admits a cellular group action by cellular homeomorphisms so that there are only finitely many orbits of i-cells for $i \le n+1$, then X has bounded geometry in dimensions $\le n+1$. If X' is a $K(G,1)$ with a finite $(n+1)$-skeleton, then its universal cover X has bounded geometry in dimensions at most equal to $n+1$.

Proposition 5.2. *Suppose that the CW-complex W has bounded geometry in dimensions up to $n+1$. Suppose that W is the union of two subcomplexes X and Y (not necessarily connected) and let $Z = X \cap Y$. Then one has the Mayer-Vietoris exact sequence*

(5.2.1)

$$\cdots \to H_i^{(1)}(Z,\mathbb{R}) \to H_i^{(1)}(X,\mathbb{R}) \oplus H_i^{(1)}(Y,\mathbb{R}) \to H_i^{(1)}(W,\mathbb{R}) \to H_{i-1}^{(1)}(Z,\mathbb{R}) \to \ldots$$

for $i \le n$.

Proof. Consider the map of pairs induced by inclusion $(Y,Z) \to (W,X)$ and corresponding commutative diagram of cellular chain complexes

(5.2.2)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_*(Z,\mathbb{R}) & \longrightarrow & C_*(Y,\mathbb{R}) & \longrightarrow & C_*(Y,Z;\mathbb{R}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{=} & & \\
0 & \longrightarrow & C_*(X,\mathbb{R}) & \longrightarrow & C_*(W,\mathbb{R}) & \longrightarrow & C_*(W,X;\mathbb{R}) & \longrightarrow & 0
\end{array}$$

We equip each of the chain groups with the ℓ_1-norm for a basis of oriented cells. For the relative chains, we use a basis of oriented i-cells of $Y - Z$ to put the ℓ_1-norm on $C_i(Y, Z; \mathbb{R})$ and the same cells to put the ℓ_1-norm on $C_i(W, X; \mathbb{R})$. Thus the vertical arrow $C_*(Y, Z; \mathbb{R}) \to C_*(W, X; \mathbb{R})$ induced by inclusion is an isomorphism of normed linear spaces. Note that X, Y, and Z have bounded geometry in dimensions at most $n+1$, and the relative chain complexes have bounded geometry, in the obvious sense, in the same range of dimensions.

Next we complete each of these vector spaces in its ℓ_1-norm topology. After completion the rows remain exact at the level of vector spaces. However the boundary maps in these chain complexes are bounded linear maps only in a range of dimensions, and in that range can be completed to give a commutative diagram of exact sequences of completed chain complexes. The crucial point is the identification of $C_i^{(1)}(Y, Z; \mathbb{R})$ with $C_i^{(1)}(W, X; \mathbb{R})$ for $i \leq n+1$, where $C_i^{(1)}(Y, Z; \mathbb{R})$ is the completion of $C_1(Y, Z; \mathbb{R})$ in its norm topology. This yields excision isomorphisms $H_i^{(1)}(Y, Z; \mathbb{R}) \cong H_i^{(1)}(W, X; \mathbb{R})$ for $i \leq n$. The argument from here on to obtain the Mayer-Vietoris exact sequence (5.2.1) is the standard one in the range of dimensions $i \leq n$.

Applications to group theory of Proposition 5.2 arise when X is a double mapping cylinder and W is a mapping torus. So suppose that $f, g : Z_1 \to X_1$ are two injective cellular maps of Z_1 into the CW-complex X_1. Let $X_2 = X_1 \cup (Z_1 \times [0, 2]) \cup (Z_1 \times [3, 5])/\sim$, where the interval $[0, 2]$ (resp. $[3, 5]$) is subdivided as $[0, 1] \cup [1, 2]$ (resp. $[3, 4] \cup [4, 5]$) and where $Z_1 \times [0, 2]$ and $Z_1 \times [3, 5]$ are given the product CW-structures. The identifications are $(z, 0) \sim f(z)$ and $(z, 5) \sim g(z)$ for all $z \in Z_1$. Let W be the quotient of X_2 after further identifications of $(z, 2)$ with $(z, 3)$ for all $z \in Z_1$, so W is the mapping torus of f, g with a particular cell structure. Let X be the image of $X_1 \cup (Z_1 \times [0, 1]) \cup (Z_1 \times [4, 5])$ in W, so X is the double mapping cylinder, and let Y be the image of $(Z_1 \times [1, 2]) \cup (Z_1 \times [3, 4])$, so Y is equivalent to Z_1. Then $Z = X \cap Y = (Z_1 \times \{1\}) \cup (Z_1 \times \{4\})$, two copies of Z_1.

Thus W is combinatorially equivalent to attaching a handle $Z_1 \times I$ to X_1 where one end of the cylinder $Z_1 \times I$ is attached by the map f and the other end is attached by the map g. The particular subdivision of the cylinder $Z_1 \times I$ was chosen above to give the Mayer-Vietoris exact sequence in an appropriate range of dimensions where the complex W has bounded geometry; this amounts to X_1 having bounded geometry in dimensions up to $(n+1)$ and Z_1 having bounded geometry in dimensions up to n.

Corollary 5.3. *Under the assumptions above, where X_1 has bounded geometry in dimensions up to $(n+1)$ and Z_1 has bounded geometry in dimensions up to n we have the exact Mayer-Vietoris sequence of ℓ_1-homology groups with real coefficients*

$$\cdots \to H_i^{(1)}(Z_1) \oplus H_i^{(1)}(Z_1) \to H_i^{(1)}(X_1) \oplus H_i^{(1)}(Z_1) \to H_i^{(1)}(W) \to$$

$$\to H_i^{(1)}(Z_1) \oplus H_i^{(1)}(Z_1) \to \dots \tag{5.3.1}$$

valid for $i \leq n$. $\mathbb{Z}$

The map $H_i^{(1)}(Z_1) \oplus H_i^{(1)}(Z_1) \to H_i^{(1)}(X_1) \oplus H_i^{(1)}(Z_1)$ is given by the matrix $\begin{pmatrix} H_i^{(1)}(f) & H_i^{(1)}(g) \\ 1 & 1 \end{pmatrix}$, where one considers the domain and ranges as column vectors. It is then an exercise to rewrite this exact sequence in the form of the next result.

Corollary 5.4. *Under the hypotheses of the previous corollary we have an exact sequence of ℓ_1-homology groups with real coefficients*

$$(5.4.1)\quad \cdots \to H_i^{(1)}(Z_1) \xrightarrow{H_i^{(1)}(f)-H_i^{(1)}(g)} H_i^{(1)}(X_1) \to H_i^{(1)}(W) \to H_{i-1}^{(1)}(Z_1) \to \dots$$

valid for $i \leq n$. $\square$

We shall apply this last result to a group G of the form $A *_C$ (resp. $A *_C B$) where A is of type F_{n+1} and C is of type F_n, $n \geq 1$ (resp. A, B of type F_{n+1} and C of type F_n). To avoid repetition we describe the HNN case, $G = A *_C$, $C < A$, $\phi : C \to A$ an injective homomorphism, so G is given by the presentation $\langle A, t \mid c^t = \phi(c), \text{ for all } c \in C\rangle$; the amalgam case $G = A *_C B$ involves only notational changes.

We can find a $K(C,1) = Z'$ with finite n-skeleton and a $K(A,1) = X'$ with finite $(n+1)$-skeleton so that the injective homomorphisms $A < C$, $\phi : C \to A$, are induced by cellular injective maps $f', g' : Z' \to X'$ respectively. Let W' be the identification complex of the double mapping cylinder of f', g' constructed above and let W be the universal cover of W'. It follows from the van Kampen theorem and Mayer-Vietoris exact sequence for singular homology that W' is a $K(G,1)$ with finite $(n+1)$-skeleton, so W is contractible with bounded geometry in dimensions up to $n+1$. Let $\widetilde{X}$, $\widetilde{Z}$ be the complete preimages of X', Z' in W under the covering map $\widetilde{W} \to W$. Then Corollary 5.4 gives an exact sequence

$$(5.4.2)\qquad \cdots \to H_i^{(1)}(\widetilde{Z}) \to H_i^{(1)}(\widetilde{X}) \to H_i^{(1)}(W) \to H_{i-1}^{(1)}(\widetilde{Z}) \to \dots$$

for $i \leq n$. The map $H_i^{(1)}(\widetilde{Z}) \to H_i^{(1)}(\widetilde{X})$ is induced by the difference of the induced maps on homology of the covering maps f, g of f', g'.

We can identify $H_i^{(1)}(W)$ with $H_i^{(1)}(G)$ for $i \leq n$ since G acts properly discontinuously, freely, and cocompactly on $W^{(n+1)}$. However in general we only get $H_i^{(1)}(C)$ as a quotient of a direct factor of $H_i^{(1)}(\widetilde{Z})$, and similarly for a relationship between $H_i^{(1)}(A)$ and $H_i^{(1)}(\widetilde{X})$. The reason is that ordinary compactly supported boundaries in these spaces are distorted in cycles, where boundaries are given the filling norm and cycles are given the induced ℓ_1-norm. This situation persists in infinite direct sums, so one cannot identify an ℓ_1-homology group of an infinite direct sum of spaces, all the same, with an ℓ_1-completion of a direct sum of ℓ_1-homology groups of the individual spaces. The phenomenon of distortion in infinite disjoint unions of spaces was explained in [Ge3] in the two paragraphs preceding Lemma 9.3 on page 1052 in the context of ℓ_∞-cohomology, and the same discussion applies to ℓ_1-homology. Hence in general the exact sequence (5.4.1) is not useful for calculating $H_i^{(1)}(C)$ and $H_i^{(1)}(A)$.

However it is shown in [Mi] that for a hyperbolic group the linear isoperimetric inequality holds for real cycles in all positive dimensions. As a consequence there is no distortion in including real boundaries in real cycles, and one can calculate the ℓ_1-homology of an infinite disjoint union of spaces all of which are isomorphic to the universal cover of an Eilenberg-MacLane space of type $K(G,1)$ of a hyperbolic group G having finite n-skeleton for all n. From this discussion follows

Proposition 5.5. *If C (resp. A) is hyperbolic then $H_i^{(1)}(\widetilde{Z}) = 0$ for all $i \geq 1$ (resp. $H_i^{(1)}(\widetilde{X}) = 0$ for all $i \geq 1$).* $\square$

An application of this result is the following combination theorem:

Theorem 5.6. *Let $G = A*_C$ (resp. $G = A *_C B$) where G and A are hyperbolic and C is finitely presented (resp. G, A, B are hyperbolic and C is finitely presented). Then C is hyperbolic.*

Proof. We argue in the HNN case, since the amalgam case involves only notational changes. Assume first that C is of type F_4. Since G and A are hyperbolic, it follows from [Mi] that $H_i^{(1)}(G) = H_i^{(1)}(A) = 0$ for all $i \geq 1$. From Proposition 5.5 it follows that $H_i^{(1)}(\widetilde{X}) = 0$ for all $i \geq 1$, and hence it follows from the exact sequence (5.4.2) that $H_i^{(1)}(\widetilde{Z}) = 0$ for all $i \geq 1$. Since $H_i^{(1)}(C)$ is a quotient group of $H_i^{(1)}(\widetilde{Z})$, it follows that $H_i^{(1)}(C) = 0$ for all $i \geq 1$. Then the main result of [AG] (which states that a finitely presented group C is hyperbolic iff $H_1^{(1)}(C, \mathbb{R}) = \bar{H}_2^{(1)}(C, \mathbb{R}) = 0$) shows that C is hyperbolic.

Now assume that C is only finitely presented. Since hyperbolic groups are of type F_∞, it follows from the Appendix below that C is of type $FP(n)$ for all n (see the Appendix, where the definition of $FP(n)$ is recalled). In particular, C is of type $FP(4)$. Since C is also finitely presented, it follows that C is of type F_4, and the argument of the preceding paragraph applies. This completes the proof.

Another consequence of these techniques is

Theorem 5.7. *Let $G = A*_C$ (resp. $G = A *_C B$) where A and C (resp. A, B, and C) are hyperbolic. Then G is hyperbolic iff the map $H_0^{(1)}(f) - H_0^{(1)}(g) : H_0^{(1)}(\widetilde{Z}) \to H_0^{(1)}(\widetilde{X})$ is injective.*

This follows from the characterization of hyperbolic groups given in [AG], (5.4.2), and Proposition 5.5.

This last result can also be rephrased in terms of distortion, as follows. Let Δ be the 1-skeleton of $\widetilde{Z}$, so Δ is the complete lift of the Cayley graph of C to W, and let Γ be the 1-skeleton of $\widetilde{X}$, so Γ can be viewed as the complete lift of the Cayley graph of A to W. Let $f, g : \Delta \to \Gamma$ denote the two inclusions corresponding to the two injective homomorphisms $C \to A$. We can assume that the images of f and g in Γ are disjoint, by replacing Γ by the 1-skeleton of the double mapping cylinder if necessary, and we assume that this is done. With these assumptions we have

Theorem 5.8. *Let $G = A*_C$ (resp. $G = A *_C B$) where A, C (resp. A, B and C) are assumed to be hyperbolic groups. Then G is hyperbolic iff the homomorphism $f_* - g_* : B_0(\Delta, \mathbb{Z}) \to B_0(\Gamma, \mathbb{Z})$ is undistorted for the filling norms on these groups.*

We defer the proof of Theorem 5.8 to the next section, since it follows immediately from Theorem 6.3.

We return now to the discussion of relative hyperbolicity in Example 4.1.1. Let H be a finitely presented subgroup of the hyperbolic group G and let Y' be a subcomplex of X', where Y', X' are spaces of type $K(H, 1), K(G, 1)$, respectively, with finite 2-skeleta such that the inclusion $Y' \subset X'$ induces $H < G$ at the π_1-level. Let X be the universal cover of X' and let Y be the complete preimage of Y' in X under the covering projection $X \to X'$. Let $\Gamma = X^{(1)}$ and let $\Delta = Y^{(1)}$, so we are in the situation of 4.1.1. Let $X//Y$ be the quotient complex of X which collapses each connected component of Y to its own vertex.

Theorem 5.9. *Under the above assumptions, the subgraph Δ of Γ is undistorted iff $X//Y$ satisfies the linear isoperimetric inequality for filling real 1-cycles of compact support. Under these circumstances, the pair (G, H) is relatively hyperbolic.*

Proof. Suppose first that Δ is undistorted in Γ, and let $\bar{w}$ be an edge-circuit in $X//Y$. Let the vertices, in order, where $\bar{w}$ meets the image of Y be $v_1, v_2, \ldots, v_n$, where we interpret the indices as integers modulo n in the following discussion. Let $\bar{A}_i$ be the portion of $\bar{w}$ between v_i and v_{i+1}, so $\bar{A}_i$ lifts to an edge-path A_i in X of the same length. Note that $\tau(A_i)$ and $\iota(A_{i+1})$ are in the same connected component of Y, so it follows that $\alpha := \sum_{i=1}^{n}(\tau(A_i) - \iota(A_{i+1}))$ is in the image of the map $B_0(\Delta, \mathbb{Z}) \to B_0(\Gamma, \mathbb{Z})$. Since Δ is undistorted in Γ, it follows that α can be filled in Y with total length bounded by $K \sum_{i=1}^{n} \text{length}(A_i)$, where K is a constant, and hence $\bar{w}$ lifts to an edge-circuit w in X of length at most $(K{+}1)\text{length}(\bar{w})$. Since G is hyperbolic, w can be filled by an integral 2-chain c in X with $|c|_1 \leq K' \text{ length}(w) \leq K'(K{+}1)\text{length}(\bar{w})$. Then c projects to an integral 2-chain $\bar{c}$ on $X//Y$ filling $\bar{w}$ with $|\bar{c}|_1 \leq K'(K+1)\text{length}(\bar{w})$, and $X//Y$ satisfies the linear isoperimetric inequality for filling integral 1-cycles. This implies the linear isoperimetric inequality for filling real 1-cycles of compact support.

Conversely, suppose the linear isoperimetric inequality holds for filling real 1-cycles on $X//Y$ of compact support. As in [AG], it follows that $H_1^{(1)}(X//Y, \mathbb{R}) = 0$. If we consider the exact sequence

$$H_1^{(1)}(X//Y, \mathbb{R}) \to I_0(Y, \mathbb{R}) \to I_0(X, \mathbb{R}),$$

which arise from the exact ℓ_1-homology sequence for the pair (X, Y) by removing the topological part $\bar{H}_0^{(1)}(Y, \mathbb{R}) \to \bar{H}_0^{(1)}(X, \mathbb{R})$ of the map $H_0^{(1)}(Y, \mathbb{R}) \to H_0^{(1)}(X, \mathbb{R})$, we see that the map $I_0(\Delta) = I_0(Y) \to I_0(X) = I_0(\Gamma)$ is injective. It follows that the map $H_0^{(1)}(\Delta, \mathbb{R}) \to H_0^{(1)}(\Gamma, \mathbb{R})$ is injective. From Theorem 4.4 it follows that the inclusion $\Delta \subset \Gamma$ is undistorted.

We suppose now that $X//Y$ satisfies the linear isoperimetric inequality for filling real 1-cycles of compact support. It follows from the methods of [Ge2] that $X//Y$ satisfies the linear isoperimetric inequality for filling integral 1-cycles. This implies by the methods of [Ge5] that its 1-skeleton $\Gamma//\Delta$ is a hyperbolic metric space, and hence that the pair (G, H) is relatively hyperbolic. This completes the proof of the theorem.

Remark. There are other situations where the pair (G, H) is relatively hyperbolic than that of the preceding theorem, as is shown by the following example.

Example 5.9.1. Let $\{x, y\}$ be a basis for $G = \mathbb{Z}^2$ and let $H = \langle x \rangle$. Then the pair (G, H) is relatively hyperbolic, since the vertex set of $\Gamma//\Delta$ is $\mathbb{Z}$ with its word metric . On the other hand, Δ is distorted in Γ; in fact Δ is the disjoint union of horizontal lines in the square lattice in $\mathbb{R}^2$ and the union of any two of these lines is distorted in $\mathbb{R}^2$ (*cf.* example 4.4.2.).

§6. Relative distortion.

Definition 6.1. Let $f, g : \Delta \to \Gamma$ be two morphisms of the graph Δ to the graph Γ. We say that f is undistorted relative to g if the induced homomorphism $f_* - g_* : B_0(\Delta, \mathbb{Z}) \to B_0(\Gamma, \mathbb{Z})$ is injective and undistorted for the filling norms on these two normed abelian groups.

Examples.

6.1.1. If $f : \Delta \to \Gamma$ is the inclusion of a subgraph and $g : \Delta \to \Gamma$ maps Δ to a vertex, then f is undistorted relative to g iff the inclusion f is undistorted according to Definition 4.1.

6.1.1. If f, g are the inclusions $\widetilde{Z}^{(1)} = \Delta \to \widetilde{X}^{(1)} = \Gamma$ in Theorem 5.8, then by that result f is undistorted relative to g iff the group G there is hyperbolic.

Remark. Since we are interested in quasi-isometry invariant properties, we can always replace Γ by the 1-skeleton of the double mapping cylinder for f, g constructed in the discussion preceding Corollary 5.3. After this replacement, we may assume that both f and g are inclusions of subgraphs with disjoint images.

Definition 6.2. If $f, g : \Delta \to \Gamma$ are morphisms of graphs, we define their *incidence graph* $I(f, g)$ as follows. A vertex of $I(f, g)$ is a connected component of Γ, and for each connected component $[v]$ of a vertex v of Δ there is an edge joining the connected components $[f(v)]$ and $[g(v)]$ of Γ. Thus the vertex set of $I(f, g)$ is $\pi_0(\Gamma)$ and geometric edges correspond 1-1 to connected components of Δ. Observe that the incidence graph is not changed by replacing Γ by the double mapping cylinder as in the remark above.

Examples.

6.2.1. Let $f : \Delta \to \Gamma$ be an arbitrary morphism of graphs and let $g : \Delta \to \pi_0(\Delta)$ be the map which collapses each connected component of Δ to its own vertex, $v \mapsto [v]$, where each edge e of Δ is collapsed to $[\iota e]$. Let $\Gamma' = \Gamma \cup \pi_0(\Delta)$ (disjoint union) and let $f', g' : \Delta \to \Gamma'$ be given by extension of range from f, g. Then $I(f', g')$ is a forest.

6.2.2. If f, g are the inclusions $\widetilde{Z}^{(1)} = \Delta \to \Gamma = \widetilde{X}^{(1)}$ in Theorem 5.8, then $I(f, g)$ is canonically isomorphic to the Bass-Serre tree associated to the HNN extension (resp. amalgam).

We come to the main result of this section.

Theorem 6.3. *Let $f, g : \Delta \to \Gamma$ be injective morphisms of graphs with disjoint images such that the incidence graph $I(f, g)$ is a forest. Then f is undistorted relative to g iff the induced homomorphism $H_0^{(1)}(f) - H_0^{(1)}(g) : H_0^{(1)}(\Delta) \to H_0^{(1)}(\Gamma)$ is injective.*

Before giving the proof, we give two examples.

Examples.

6.3.1. Suppose $f : \Delta \to \Gamma$ is the inclusion of a subgraph and let Γ', f', g' be as in example 6.2.1. One checks that the kernel of $H_0^{(1)}(f') - H_0^{(1)}(g')$ is exactly the kernel of $I_0(f) : I_0(\Delta) \to I_0(\Gamma)$, so Theorem 6.3 reduces in this situation to Theorem 4.4.

6.3.2. If f, g are the inclusions $\widetilde{Z} = \Delta \to \widetilde{X} = \Gamma$ in Theorem 5.8, then since $I(f, g)$ is a tree, Theorem 6.3 reduces in this case to Theorem 5.8.

Proof of Theorem 6.3. If we assume that $H_0^{(1)}(f) - H_0^{(1)}(g)$ is injective, then the argument proceeds with no changes from the argument for the converse direction of Theorem 3.2 to show that $f_* - g_* : B_0(\Delta, \mathbb{Z}) \to B_0(\Gamma, \mathbb{Z})$ is undistorted. So we shall assume now that $f_* - g_* : B_0(\Delta, \mathbb{Z}) \to B_0(\Gamma, \mathbb{Z})$ is undistorted and prove that $H_0^{(1)}(f) - H_0^{(1)}(g)$ is injective.

Let $\alpha \in C_0^{(1)}(\Delta)$ be such that there exists $\beta \in C_1^{(1)}(\Gamma)$ with $f_*(\alpha) - g_*(\alpha) = \hat{\partial}\beta$. Let $p : \Gamma \to \bar{\Gamma}$ be the map which identifies $f(v)$ with $g(v)$ for each vertex v of Δ, so p is bijective on edges; thus 1-chains on $\bar{\Gamma}$ lift uniquely to 1-chains on Γ. Then $p_*(\alpha) = 0$, so $p_*(\beta) \in Z_1^{(1)}(\Gamma, \mathbb{R})$. Thus we can apply the approximation theorem [AG] to $p_*(\beta)$ and write it as a coherent positive sum $p_*(\beta) = \sum_j u_j z_j$, $u_j > 0$, where z_j are simple circuits on $\bar{\Gamma}$. Lift z_j back to Γ and let ζ_j be the integral 1-chain on Γ so determined. We need

Lemma 6.4. *$\partial\zeta_j \in \mathrm{Image}(f_* - g_* : B_0(\Delta, \mathbb{Z}) \to B_0(\Gamma, \mathbb{Z}))$.*

The lemma follows from the assumption that the incidence graph $I(f, g)$ is a forest; the circuit z_j determines a circuit in $I(f, g)$, and such a circuit must have a backtrack. Removing this backtrack and proceding by induction enables us to construct the element of $\mathrm{Image}(f_* - g_* : B_0(\Delta, \mathbb{Z}) \to B_0(\Gamma, \mathbb{Z}))$ in the lemma.

Since by assumption the map $f_* - g_* : B_0(\Delta, \mathbb{Z}) \to B_0(\Gamma, \mathbb{Z}))$ is undistorted, $\zeta_j = \partial(f_*(\beta'_j) - g_*(\beta'_j))$ with $\beta'_j \in C_1(\Delta, \mathbb{Z})$ and $|\beta'_j|_1 \leq K|z_j|_1$ for a constant K independent of j. It follows that $\beta' := \sum_j u_j \beta'_j \in C_1^{(1)}(\Delta, \mathbb{R})$ and $\hat{\partial}\beta' = \alpha$; this last step uses injectivity of $f_* - g_*$ at the chain level, which in turn follows since the images of f and g are disjoint. This completes the proof of the theorem.

This last result has a geometric consequence.

Theorem 6.5. *Let $f, g : \Delta \to \Gamma$ be injective morphisms of graphs with disjoint images such that the incidence graph $I(f, g)$ is a forest. If f is undistorted relative to g, then the inclusion $\Gamma_f \cup \Gamma_g \subset \Delta \times \Gamma$ is undistorted, where Γ_f and Γ_g are the graphs of the maps f, g respectively.*

Before giving the proof, we make some remarks about the product of graphs $\Delta \times \Gamma$ occurring in the statement. Of course if these graphs are viewed as CW complexes of dimension 1, then the product is a 2-complex. We could use the 1-skeleton of this 2-complex except that there is no canonical way of interpreting the graph Γ_f (it is the same difficulty that there is no canonical cellular approximation to the diagonal in CW-complexes). To get around this problem one could make choices to define Γ_f, or one could proceed more elegantly using a device introduced in [Ge7], redefining what one means by a graph.

A graph X is a set with involution $x \mapsto \bar{x}$, $x \in X$, together with a retraction $\iota : X \to V(X)$, where $V(X)$ is the set of fixed points of the involution. A morphism of graphs is a map of sets which commutes with the involutions and with the retractions. One interprets $V(X)$ as the set of vertices of X, the map ι is the initial vertex, and $\tau(x) = \iota(\bar{x})$ is the terminal vertex. An edge $x \in X$ is an element such that $x \neq \bar{x}$. Then this category of graphs admits products and admits a geometric realization.

For example, the analog X of the interval in this category has 4 elements, $0, 1, x, \bar{x} \neq x$, where 0 and 1 are vertices, $\iota x = 0$ and $\iota\bar{x} = 1$. Its geometric realization is the unit interval. The geometric realization of $X \times X$ is the 1-skeleton of the square with the two diagonals adjoined disjointly, namely, the complete graph on four vertices. It is these diagonals which permit diagonal approximation.

Having addressed the question what one means by Γ_f, we proceed to the

Proof of Theorem 6.5. It follows from Theorem 6.3 that $H_0^{(1)}(f) - H_0^{(1)}(g)$: $H_0^{(1)}(\Delta) \to H_0^{(1)}(\Gamma)$ is injective.

Let Δ_i be disjoint copies of Δ, $i = 1, 2$ and let $\nabla : \Delta_1 \cup \Delta_2 \to \Delta \times \Gamma$ be equal to the graph of f when restricted to Δ_1 and equal to the graph of g when restricted to Δ_2, so, $\nabla(x) = (x, f(x))$ for $x \in \Delta_1$ and $\nabla(x) = (x, g(x))$ for $x \in \Delta_2$. One has $H_0^{(1)}(\Delta_1 \cup \Delta_2) = H_0^{(1)}(\Delta) \oplus H_0^{(1)}(\Delta)$. One has also projections from $\Delta \times \Gamma$ to Δ and to Γ inducing the map $p : H_0^{(1)}(\Delta \times \Gamma) \to H_0^{(1)}(\Delta) \oplus H_0^{(1)}(\Gamma)$ so that $p \circ H_0^{(1)}(\nabla)(\xi_1, \xi_2) = (\xi_1 + \xi_2, H_0^{(1)}(f)(\xi_1) + H_0^{(1)}(g)(\xi_2))$ for $\xi_1, \xi_2 \in H_0^{(1)}(\Delta)$.

Suppose that (ξ_1, ξ_2) is in the kernel of $H_0^{(1)}(\nabla)$. It follows that $p \circ H_0^{(1)}(\nabla)(\xi_1, \xi_2) = (0, 0)$, so $\xi_1 + \xi_2 = 0$ and $H_0^{(1)}(f)(\xi_1) + H_0^{(1)}(g)(\xi_2) = 0$. Thus $\xi_2 = -\xi_1$ and $H_0^{(1)}(f)(\xi_1) = H_0^{(1)}(g)(\xi_1)$. That is, $\xi_1 \in \mathrm{Ker}(H_0^{(1)}(f) - H_0^{(1)}(g))$. It follows that $\xi_1 = 0$, so $(\xi_1, \xi_2) = (0, 0)$.

Thus $H_0^{(1)}(\nabla)$ is injective, and Corollary 3.9 shows that the inclusion of the image of ∇ in $\Delta \times \Gamma$ is undistorted. That is, $\Gamma_f \cup \Gamma_g$ is undistorted in $\Delta \times \Gamma$, and the proof is complete.

Corollary 6.6. *If $G = A*_C$ (resp. $G = A*_C B$) where all the groups are hyperbolic, and if $f, g : \Delta \to \Gamma$ are as in Theorem 5.8, then the inclusion $\Gamma_f \cup \Gamma_g \subset \Delta \times \Gamma$ is undistorted, where Γ_f and Γ_g are the graphs of the maps f and g respectively.* $\mathbb{Z}$

6.7. Open questions.

6.7.1. I do not know if the conclusion of Theorem 6.5, *i.e.* the condition that the union of the graphs of f and g be distorted in $\Delta \times \Gamma$, is equivalent to the condition that f be undistorted relative to g.

Remark. If $I_0(\Delta \times \Gamma)$ were equal to $I_0(\Delta) \oplus I_0(\Gamma)$ via projections to the two factors, then an easy argument would show that question 6.7.1 had an affirmative answer. However $I_0(\Delta \times \Gamma) \neq I_0(\Delta) \oplus I_0(\Gamma)$ when Δ and Γ are both equal to the Cayley graph of $\mathbb{Z}$. In this case $\Delta \times \Gamma$ is the unit square lattice in the first quadrant in the plane. We let $b_n = (n, n) - (n, 0) - (0, n) + (0, 0)$, and note that $b = \sum_{n \geq 1} b_n / n^2 \in C_0^{(1)}(\mathbb{Z} \times \mathbb{Z}, \mathbb{R})$. Also $|b_n|_{\mathbb{Z} \times \mathbb{Z} \mathrm{fill}} = 2n$, and it can be proved that $|b|_{\mathbb{Z} \times \mathbb{Z} \mathrm{fill}} = \sum 2n/n^2 = \infty$, so $b \notin B_0^{(1)}(\mathbb{Z} \times \mathbb{Z}, \mathbb{R})$ (*e.g.*, argue by contradiction, project along lines of slope 1 to the union U of the positive x-and y-axes, and reduce to a 1-dimensional problem for the tree U). However clearly b_n projects to 0 in both factors, so b projects to 0 in $I_0(\mathbb{Z}) \oplus I_0(\mathbb{Z})$.

This calculation shows that there is no obvious analog for the Künneth formula in this homology theory.

6.7.2. What additional hypotheses on $f, g : \Delta \to \Gamma$ does one need to assume to guarantee that a lack of distortion for $f_* - g_*$ on unramified boundaries implies that $f_* - g_*$ is unramified on all of $B_0(\Delta, \mathbb{Z})$?

Remark. If $f, g : \Delta \to \Gamma$ are as in Theorem 5.8, a version of the combination theorem of [BF] states that under the hypotheses that A, C (resp. A, B, C) are hyperbolic and both inclusions of C in A (resp. the inclusions of C in A and B) are undistorted, a sufficient condition that f be undistorted relative to g is that $f_* - g_* : B_0(\Delta, \mathbb{Z}) \to B_0(\Gamma, \mathbb{Z})$ is uniformly undistorted when restricted to unramified 0-boundaries. In fact, in the terminology of [BF], one need only check that 0-boundaries on Δ arising from essential hallways are undistorted under $f_* - g_*$ to assure hyperbolicity of $G = A*_C$ (resp. $G = A*_C B$).

6.7.3. Does there exist a group G of type F_{n+1} for some $n \geq 2$ for which the *reduced* ℓ_1-homology group $\bar{H}_n^{(1)}(G, \mathbb{R}) \neq 0$? It is shown in [Mi] that if G is combable in the sense of [EC], then $\bar{H}_n^{(1)}(G, \mathbb{R}) = 0$ for all $n \geq 0$; this includes all hyperbolic and automatic groups.

Appendix A.

The purpose of this appendix is to prove Theorem A1 below, which was used in the proof of Theorem 5.6 above.

Theorem A1. *If $G = A *_C B$ (resp. $G = A*_C$) where G is of type $FP(n)$ and A and B are both of type $FP(n-1)$ (resp. G is of type $FP(n)$ and A is of type $FP(n-1)$) for some $n \geq 2$) and where C is finitely generated, then C is of type $FP(n-1)$.*

Recall that the group G is of type $FP(n)$ if the trivial G-module $\mathbb{Z}$ has a partial resolution

$$P_n \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0,$$

where P_i are finitely generated free $\mathbb{Z}G$-modules, $0 \leq i \leq n$. If G is of type F_n, then it is of type $FP(n)$, and G is of type F_1 iff it is of type $FP(1)$ iff it is finitely generated [Bi]. Furthermore, if G is of type $FP(n)$ and G is finitely presented, then G is of type F_n [Br].

The following criterion appears in [BE] Prop. 1.2; to make the translation from their statement, note that $\mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, M) = H_i(G, M)$ and note that $\mathbb{Z}$ is a finitely presented $\mathbb{Z}G$-module if G is a finitely generated group:

Theorem A2 (*cf.* [BE] Prop. 1.2)**.** *If G is a finitely generated group, then G is of type $FP(n)$ for some $n \geq 2$ iff $H_i(G, \prod_J \mathbb{Z}G) = 0$ for $1 \leq i \leq n-1$ and all direct products $\prod_J \mathbb{Z}G$ for arbitrary index sets J.*

We also need the following result which follows from [Bi] Theorem 1.3 (i⇒iia).

Theorem A3 (*cf.* [Bi] Theorem 1.3)**.** *If G is of type $FP(n)$ then $H_i(G, \prod_J M) = \prod_J H_i(G, M)$ for $1 \leq i \leq n-1$ for all $\mathbb{Z}G$-modules M and all direct products $\prod_J M$.*

Proof of Theorem A1. We argue in the amalgam case, since the HNN case requires only notational changes. Since C is finitely generated, to prove that C is of type $FP(n-1)$ it suffices by Theorem A2 to show that $H_i(C, \prod_J \mathbb{Z}C) = 0$ for $1 \leq i \leq n-2$ and all J. Since $C < G$, $\prod_J \mathbb{Z}C$ is a retract of $\prod_J \mathbb{Z}G$ as $\mathbb{Z}C$-modules (this follows since $\mathbb{Z}G$ is a free $\mathbb{Z}C$-module). So it suffices to prove that $H_i(C, \prod_J \mathbb{Z}G) = 0$ for $1 \leq i \leq n-2$ and all J.

From the Mayer-Vietoris exact sequence for an amalgam, [Br] p. 178 (9.1), and the fact that $H_i(G, \prod_J \mathbb{Z}G) = 0$ for $1 \leq i \leq n-1$, we see that it suffices to prove that $H_i(A, \prod_J \mathbb{Z}G) = H_i(B, \prod_J \mathbb{Z}G) = 0$ for $1 \leq i \leq n-2$. But A and B are of type $FP(n-1)$, so by Theorem A3 we have $H_i(A, \prod_J \mathbb{Z}G) = \prod_J H_i(A, \mathbb{Z}G)$, $1 \leq i \leq n-2$, and similarly for B. Also $H_i(A, \mathbb{Z}G) = 0$ for $i > 0$ since $\mathbb{Z}G$ is a free $\mathbb{Z}A$-module, so it follows that $H_i(A, \prod_J \mathbb{Z}G) = 0$ for $1 \leq i \leq n-2$, and similarly for B. Thus $H_i(C, \prod_J \mathbb{Z}G) = 0$ for $1 \leq i \leq n-2$, and it follows that C is of type $FP(n-1)$. This completes the proof.

Appendix B.

At the Pusan conference A. Yu. Ol'shanskii asked me if there was a homological interpretation of the distortion function f (see the Remark following Definition 3.1) associated to Cayley graphs of subgroup and group. This appendix sketches a theory leading to such an interpretation. I mention that this is part of a program, as yet unfulfilled, whose goal is to give homological interpretations for the notions of geometric group theory, including the Dehn function and the boundary of a (hyperbolic) group; so far, only hyperbolicity [Ge2][AG] and distortion, treated here, have been shown to admit such interpretations.

Definition B1. A metric graph Γ assigns to each geometric edge a positive real number called its weight w; equivalently, if we interpret Γ as a Serre graph, we have an assignment $e \mapsto w(e) > 0$ such that $w(\bar{e}) = w(e)$, where $\bar{e}$ is the oppositely oriented edge to e. The metric graph determines a path metric on Γ which in general does not agree with the given assignment of weights on the edges. If the path metric agrees with w on edges, then we say that w is a *consistent* assignment of weights.

Examples.

B1.1. Let Γ be a triangle with weights 1,1, and 3 assigned to the geometric edges. In the associated path metric the length of the long edge is 2, not 3.

B1.2. If all geometric edges of the connected graph Γ are assigned weight 1, then the associated path metric is the "word metric" d_Γ, and two adjacent vertices have distance 1.

B1.3. If Γ is a tree, then all assignments of weights to edges are consistent.

Now let Γ be a connected graph equipped with the word metric B1.2., and let $C(\Gamma)$ be the disjoint union of Γ with the edges of complete graph on the same set of vertices. We call the edges of $C(\Gamma) \square \Gamma$ *new edges*. If e is an oriented new edge joining vertex v to vertex w, then we assign to e the weight $d_\Gamma(v, w)$, while all edges of Γ are assigned weight 1.

Lemma B2. *The assignment of weights on $C(\Gamma)$ just defined is consistent, and the associated metric agrees with the word metric on the vertices of Γ.* $\square$

Now suppose that Δ is a connected subgraph of the connected graph Γ. Then the word metric d_Γ determines a weight on the edges of $C(\Delta)$. We call this metric graph $D(\Delta)$ (it is the same underlying graph as $C(\Delta)$ with different weights attached to the edges), and there are natural injective maps of metric graphs $C(\Delta) \to D(\Delta) \to C(\Gamma)$ which do not increase weights (note however that $D(\Delta)$ is not in general consistent). In addition, the inclusion $D(\Delta) \subset C(\Gamma)$ is undistorted in the obvious sense[6], since the weights on edges of $D(\Delta)$ are determined by restriction from those of $C(\Gamma)$.

We can apply this to the Cayley graphs of a finitely generated subgroup H of a finitely generated group G. Let B be a finite set of generators of G containing a subset A of generators for H, and let Δ be the associated Cayley graph of H considered as a subgraph of the Cayley graph Γ of G. Then we have the injective

[6] A connected metric graph X determines a pseudometric by taking the infimum of weighted lengths of paths. If Y is a connected subgraph of X equipped with its own weight on edges, then Y is said to be undistorted in X if (1) the X-weights on edges of Y are at most equal to their Y-weights ("the inclusion $Y \subset X$ is weight nonincreasing") and (2) the pseudometric determined by the weight of Y is bilipschitz equivalent to the restriction of that determined by X on vertices of Y.

maps $C(\Delta) \to D(\Delta) \to C(\Gamma)$ with the second map undistorted. The next result is a consequence of the cocompactness of the group actions on the respective Cayley graphs.

Lemma B3. *The distortion function f associated to the Cayley graphs Γ, Δ is given by the rule that $f(n)$ is equal to the maximum over all new edges e of $C(\Delta)$ of weight at most n in $D(\Delta)$ of their weight in $C(\Delta)$.* $\mathbb{Z}$

It is also possible to formulate this homologically. If Γ is a metric graph with weight w on the edges, then we equip $C_1(\Gamma, \mathbb{R})$ with the norm $|c| = \sum_e |a_e| w(e)$ if $c = \sum_e a_e e$, where e runs over geometric edges in both summations and where the coefficients $a_e = 0$ for all but finitely many edges e. Then the boundary map $\partial : C_1(\Gamma, \mathbb{R}) \to C_0(\Gamma, \mathbb{R})$ is continuous if the weights of edges are bounded below by a positive constant (as will always be the case in Cayley graphs of finitely generated groups) and in this case induces a map of completions $\hat{\partial} : C_1^{(1)}(\Gamma, \mathbb{R}) \to C_0^{(1)}(\Gamma, \mathbb{R})$ with kernel $Z_1^{(1)}(\Gamma, \mathbb{R})$ and cokernel $H_0^{(1)}(\Gamma, \mathbb{R})$. Here $C_1^{(1)}$ consists of sums $c = \sum_e a_e e$ such that $\sum_e |a_e| w(e)$ converges, and one has the exact sequence

$$0 \to Z_1^{(1)}(\Gamma, \mathbb{R}) \to C_1^{(1)}(\Gamma, \mathbb{R}) \to C_0^{(1)}(\Gamma, \mathbb{R}) \to H_0^{(1)}(\Gamma, \mathbb{R}) \to 0.$$

Both $Z_1^{(1)}$ and $H_0^{(1)}$ are functors from the category of metric graphs and morphisms of graphs which are weight non-increasing to that of real vector spaces. In the situation of Lemma B2 this means that we have induced homomorphisms $H_0^{(1)}(C(\Delta), \mathbb{R}) \to H_0^{(1)}(D(\Delta), \mathbb{R}) \to H_0^{(1)}(C(\Gamma), \mathbb{R})$. In addition the approximation theorem holds for $Z_1^{(1)}$, so such cycles can be approximated by cycles of compact support, and every such cycle can be represented as a convergent coherent sum of simple cycles for the weighted norm topology of C_1. Since $D(\Delta) \to C(\Gamma)$ is undistorted, the analog of Theorem 3.2 holds to show that the induced homomorphism $H_0^{(1)}(D(\Delta), \mathbb{R}) \to H_0^{(1)}(C(\Gamma), \mathbb{R})$ is injective, so the kernel of the map $H_0^{(1)}(C(\Delta), \mathbb{R}) \to H_0^{(1)}(C(\Gamma), \mathbb{R})$ is equal to the kernel of the map $H_0^{(1)}(C(\Delta), \mathbb{R}) \to H_0^{(1)}(D(\Delta), \mathbb{R})$. Furthermore since the map $H_0^{(1)}(C(\Delta), \mathbb{R}) \to H_0^{(1)}(D(\Delta), \mathbb{R})$ is surjective, it follows that $H_0^{(1)}(C(\Delta), \mathbb{R})$ $\mathbb{Z}$ $H_0^{(1)}(D(\Delta), \mathbb{R})$ $\mathbb{R}$ $H_0^{(1)}(C(\Gamma), \mathbb{R})$ is the canonical factorization of the map $H_0^{(1)}(C(\Delta), \mathbb{R}) \to H_0^{(1)}(C(\Gamma), \mathbb{R})$, thereby characterizing $H_0^{(1)}(D(\Delta), \mathbb{R})$ as a quotient of $H_0^{(1)}(C(\Delta), \mathbb{R})$.

Now $C_1^{(1)}(D(\Delta), \mathbb{R})$ is precisely the set of those chains in $C_1^{(1)}(C(\Gamma), \mathbb{R})$ which are supported on the edges of $C(\Delta)$. Let w denote the weight on the edges of $D(\Delta)$ obtained by restricting the weight of $C(\Gamma)$.

Lemma B4. *The weight w is determned up to bilipschitz equivalence by the normed linear space $C_1^{(1)}(D(\Delta), \mathbb{R})$.*

Proof. Let w' be another weight on the edges of $D(\Delta)$ such that $C_{1,w'}^{(1)}(D(\Delta), \mathbb{R}) := \{\sum_e a_e e \mid \sum_e |a_e| w'(e) < \infty\} = C_1^{(1)}(D(\Delta), \mathbb{R})$. Let $|\ \ |_{w'}$ denote the norm on $C_{1,w'}^{(1)}(D(\Delta), \mathbb{R})$ and let $|\ \ |_w$ denote the norm on $C_1^{(1)}(D(\Delta), \mathbb{R})$. Suppose that $\{w(e)/w'(e) \mid e \text{ edge of } D(\Delta)\}$ is an unbounded set of positive real numbers. Then there exists a sequence of edges e_i so that $w(e_i)/w'(e_i) > 2^i$ for $i \in \mathbb{Z}$. If $c = \sum_i \dfrac{e_i}{2^i w'(e_i)}$, then $|c|_{w'} = \sum_i 1/2^i < \infty$, so $c \in C_{1,w'}^{(1)}(C(\Delta), \mathbb{R})$. But $|c|_w = \sum_i \dfrac{w(e_i)}{2^i w'(e_i)} > \sum_i 2^i/2^i = \infty$, so $c \notin C_1^{(1)}(D(\Delta), \mathbb{R})$, contrary to hypothesis.

Similarly one shows that $\{w'(e)/w(e) \mid e \text{ edge of } D(\Delta)\}$ is bounded, and the lemma is established.

Summarizing the discussion, we have

Theorem B5. *If Δ is a connected subgraph of the connected graph Γ, then the induced inclusion of consistent metric graphs $C(\Delta) \to C(\Gamma)$ factors as $C(\Delta) \to D(\Delta) \to C(\Gamma)$, where the first arrow is a bijection on edges and the second is undistorted. The induced maps on $H_0^{(1)}$ (as defined above in the category of metric graphs) give the canonical factorization of the map $H_0^{(1)}(C(\Delta, \mathbb{R}) \to H_0^{(1)}(C(\Gamma), \mathbb{R})$.*

Furthermore Lemma B3 shows how the distortion function can be recovered from the map $C(\Delta) \to D(\Delta)$ in the case of Cayley graphs of finitely generated groups.

Remark. Theorem B5 does not give a complete answer to Ol'shankii's question of characterizing the distortion function homologically. To do that, one must prove the full analogs of Theorem 3.2 and Corollary 3.8 for metric graphs, in addition to a version of [Ge4] Theorem 9.2 (on the regularity of the integral filling norm for 0-boundaries) adapted to metric graphs. All this is true under the hypothesis of positive integral weights, and we hope to treat it in a future article.

Question. Is it possible to give a (co-)homological interpretation for the Dehn function of a finitely presented group? There are three fundamental difficulties here I have not yet been able to overcome. First, one does not know in general whether the Dehn function and its homological analog (involving fillings by orientable surface diagrams of arbitrary genus) are equivalent. Second, there is no analog of the approximation theorem [AG] for summable 2-cycles in general; one possible way to handle this is to deal only with ℓ_∞-cohomology. Third, the integral filling norm for 1-dimensional boundaries is not in general regular, and the best one can expect (conjecturally) is that the real and integral filling norms for integral 1-cycles are bilipschitz equivalent on the universal cover of the (finite) presentation complex; this question is open even for $\mathbb{Z}^4$ [Ge4].

Appendix C.

As I remarked in the Introduction, the starting point for this paper was my result that a finitely generated subgroup H of the hyperbolic group G is quasi-isometrically imbedded (and hence H is quasi-convex in G) iff the restriction homomorphism $H^1_{(\infty)}(G, \mathbb{Z}) \to H^1_{(\infty)}(H, \mathbb{Z})$ is surjective [Ge3]. The purpose of this appendix is to prove an analogous result for arbitrary finitely generated groups:

Theorem C1. *If H is a finitely generated subgroup of the finitely generated group G, then the inclusion $H < G$ is undistorted for their respective word metrics (i.e. $H < G$ is a quasi-isometric imbedding) iff the restriction homomorphism $H^1_{(\infty)}(G, \ell_\infty) \to H^1_{(\infty)}(H, \ell_\infty)$ is surjective.*

Remark. Since there is no exact duality between ℓ_∞ and ℓ_1 (for although ℓ_∞ is the dual of $\ell_!$, the dual of ℓ_∞ is not ℓ_1), Theorem C1 appears to require additional arguments beyond Corollary 3.8 (the last result will however be used in the proof that follows). Also, although the surjectivity of the restriction homomorphism with ℓ_∞-coefficients in Theorem C1 implies the surjectivity with $\mathbb{R}$-coefficients, I do not know whether the converse holds in general (it does hold for hyperbolic groups by [Ge3]).

Proof of Theorem C1. Let X' be a $K(G,1)$ with finite 1-skeleton containing as subcomplex Y', a $K(H,1)$, so that the inclusion $Y' \subset X'$ induces for appropriate choice of base point in Y' the inclusion homomorphism $H < G$ at the fundamental group level. Let X be the universal cover of X' and let Y be a connected component of the pull-back of Y' to X, so Y is the universal cover of Y'. The inclusion $Y \subset X$ induces the restriction homomorphism $H^1_{(\infty)}(X,A) \to H^1_{(\infty)}(Y,A)$ for any normed abelian group A.

By Corollary 3.8 above, $H < G$ is undistorted iff the homomorphism $i_* : B_0(Y,\mathbb{R}) \to B_0(X,\mathbb{R})$ induced by inclusion $Y \subset X$ is undistorted as a map of normed linear spaces over the reals. Here the boundaries are equipped with their filling norms, so one has for example the short exact sequence

$$0 \to Z_1(Y,\mathbb{R}) \to C_1(Y,\mathbb{R}) \to B_0(Y,\mathbb{R}) \to 0,$$

where $B_0(Y,\mathbb{R})$ is given the quotient norm induced from the ℓ_1-norm on $C_1(Y,\mathbb{R})$; similar statements hold for X.

Now if A is a normed linear space over $\mathbb{R}$, then it follows from the fact that $B_1(Y,\mathbb{R}) = Z_1(Y,\mathbb{R})$ that $Z^1_{(\infty)}(Y,A) = \mathrm{Hom}_c(B_0(Y,\mathbb{R}),A)$, where the right side denotes the set of continuous linear maps from $B_0(Y,\mathbb{R})$ to A. In addition $B^1_{(\infty)}(Y,A)$ consists of those maps $f \in \mathrm{Hom}_c(B_0(Y,\mathbb{R}),A)$ for which there exists $g \in \mathrm{Hom}_c(C_0(Y,\mathbb{R}),A)$ whose restriction to $B_0(Y,\mathbb{R})$ is equal to f. We need the following

Lemma C2. *Let $i : V \to W$ be a continuous injective homomorphism of normed linear spaces over $\mathbb{R}$. Then i is undistorted iff for every continuous linear map $f : V \to \ell_\infty$ there exists a continuous linear map $g : W \to \ell_\infty$ such that $g \circ i = f$.*

Proof. Suppose first that $i : V \to W$ is undistorted, and let $f : V \to \ell_\infty$ be a continuous linear map. Then f is the same as a collection of linear functionals $f_j : V \to \mathbb{R}$ with $|f_j| \le K$ for all $j \ge 0$, where $|f_j|$ denotes the norm of f_j, namely $|f_j| = \sup_{\substack{|v|=1 \\ v\in V}} |f_j(v)|$. By the Hahn-Banach theorem there exists $g_j : W \to \mathbb{R}$ with $|g_j| \le K'|f_j|$ for each j, where K' is a constant independent of j, so the collection $\{g_j,\ j \ge 0\}$ defines $g : W \to \ell_\infty$ with $|g| \le KK'$ such that $g \circ i = f$.

Conversely, suppose that every $f : V \to \ell_\infty$ extends continuously to $g : W \to \ell_\infty$. We argue by contradiction, assuming that $i : V \to W$ is distorted. This means that there exist $v_j \in V$ with $|v_j| = 1$ and $|i(v_j)| \le 1/2^j$ for each $j \ge 0$. Now the Hahn-Banach theorem guarantees there are continuous linear maps $f_j : V \to \mathbb{R}$ with $f_j(v_j) = 1$ and $|f_j| = 1$ for all $j \ge 0$. These maps give rise to $f : V \to \ell_\infty$ with $|f| = 1$, and by hypothesis f extends to give $g : W \to \ell_\infty$ with g continuous and linear. Let $g_j : W \to \mathbb{R}$ be the j-th coordinate function of g, so $g_j \circ i = f_j$. Now $|g(i(v_j))| = \sup_k |g_k(i((v_j))| \ge |g_j(i(v_j))| = |f_j(v_j)| = 1$ for all j. But $|i(v_j)| \to 0$ as $j \to \infty$ by the continuity of the map i. This contradicts the continuity of g, and it follows that $i : V \to W$ is undistorted.

Continuing with the proof of Theorem C1, suppose that $H < G$ is undistorted, so $B_0(Y,\mathbb{R}) \to B_0(X,\mathbb{R})$ is undistorted. By the Lemma, every $f \in \mathrm{Hom}_c(B_0(Y,\mathbb{R}),\ell_\infty)$ is the restriction of some $g \in \mathrm{Hom}_c(C_0(X,\mathbb{R}),\ell_\infty)$. Note also that if $f : C_0(Y,\mathbb{R}) \to \ell_\infty$ is continuous and linear, then f extends continuously to $C_0(X,\ell_\infty)$ since $C_0(Y,\ell_\infty)$ is a continuous retract of $C_0(X,\ell_\infty)$. It follows from these observations that the restriction homomorphism $H^1_{(\infty)}(X, \ell_\infty) \to H^1_{(\infty)}(Y, \ell_\infty)$ is surjective.

For the converse assume that the restriction homomorphism $H^1_{(\infty)}(X, \ell_\infty) \to H^1_{(\infty)}(Y, \ell_\infty)$ is surjective. By the preceding paragraph it follows that every continuous linear map $f : B_0(Y, \mathbb{R}) \to \ell_\infty$ is the restriction of a continuous linear map $g : B_0(X, \mathbb{R}) \to \ell_\infty$. It follows from the Lemma that the inclusion $B_0(Y, \mathbb{R}) \to B_0(X, \mathbb{R})$ is undistorted, and hence $H < G$ is undistorted by Corollary 3.8. This completes the proof of the Theorem.

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Mathematics Department, University of Utah, Salt Lake City, UT 84112, USA
E-mail address: gersten@math.utah.edu