Some isoperimetric inequalities for kernels of free extensions

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Abstract. If G is a hyperbolic group (resp. synchronously or asynchronously automatic group) which can be expressed as an extension of a finitely presented group H by a finitely generated free group, then the normal subgroup H satisfies a polynomial isoperimetric inequality (resp. exponential isoperimetric inequality).

Keywords: finitely presented group, word metric, van Kampen diagram, radius, area, isoperimetric function, hyperbolic group, automatic group

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For John Stallings on his 65th birthday.

1. Introduction

There is an extensive literature on hyperbolic groups, beginning with Gromov's fundamental paper [13] and its exegeses, for instance [12], [6], [1], [2]. However, remarkably little is known about subgroups of general hyperbolic groups; for example, it is unknown how distorted the word metrics of finitely generated subgroups can be and how distorted the areas of finitely presented subgroups [9] can be.[1] Recall that, for a given finite presentation, the area of a relation is the minimum number of terms expressing the relation as a product of conjugates of relators. An isoperimetric function (or isoperimetric inequality) for a given finite

[1] An open test question is whether a finitely generated subgroup of a hyperbolic group can be distorted more than exponentially in its word metric. Is even exponential distortion possible in area for a finitely presented subgroup? We mention in this context a remarkable recent result of P. Papasoglu that the area distortion function of a finitely presented subgroup of a finitely presented group is *always* recursive [18].

presentation gives an upper bound for the area of the relations as a function of their length. The growth of this function is an invariant of the group (see below), and a group is hyperbolic if and only if this growth is linearly bounded.

Here is a brief sketch of what is known about such subgroups. A finitely generated subgroup of a hyperbolic group G has a solvable word problem (since G itself has a solvable word problem) and a subgroup of G has finite rational cohomological dimension (since G itself has finite rational cohomological dimension). From the action of G on the Gromov boundary one knows that solvable (and more generally amenable) subgroups of G are virtually cyclic, and if a subgroup is not virtually solvable then it contains a nonabelian free subgroup [13]. Furthermore Rips showed there can exist finitely generated subgroups of hyperbolic groups which are not finitely presentable [20] (*i.e.* in general hyperbolic groups are not *coherent*). N. Brady gave one example of a hyperbolic group $G = H \rtimes_\phi \mathbb{Z}$ where H is finitely presented but not of type FP_3 [3]; since hyperbolic groups are of type FP_∞, H is not hyperbolic. In his example, G is of cohomological dimension 3; this contrasts with the result of [11] that a finitely presented subgroup of a hyperbolic group of cohomological dimension 2 is hyperbolic. Finally, there is a universal bound on the order of a finite subgroup of a hyperbolic group G that depends only on the number of generators and the Rips constant δ for that set of generators [4].

The main application of our general result is the following:

THEOREM A.

Let G be a split extension of a finitely presented group K by a finitely generated free group F, so one has the short exact sequence

$$1 \to K \to G \to F \to 1.$$

(1) If G is hyperbolic, then K satisfies a polynomial isoperimetric inequality.
(2) If G is (either synchronously or asynchronously) automatic, then K satisfies an exponential isoperimetric inequality.

It follows from conclusion (1) that in Brady's example of the preceding paragraph, H satisfies a polynomial isoperimetric inequality. This answers a question raised in [3], and improves the result of [9], where the first author showed that H had an exponential isoperimetric function.

Concerning conclusion (2), there is an example due to M. Bridson *et al* (see [5]), which was discussed in [9] section 4, of a homomorphism of a synchronously automatic group onto $\mathbb{Z}$ with kernel finitely presentable and having an optimal exponential isoperimetric function (*i.e.*

the Dehn function of the kernel is exponential). Thus conclusion (2) is best possible in this generality.

To prove Theorem A, we use the diagrammatic methods of [9] with some important minor adjustments to obtain a more general result. Consider a split extension $1 \to K \to G \to F \to 1$ with K finitely presented and F a finitely generated free group. We obtain an isoperimetric inequality for K in terms of a combined area/radius function (an "AR" pair) for G. The precise result is stated below as Theorem B.

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2. Isoperimetric functions and AR pairs

Let $\mathcal{P} = \langle X \mid R \rangle$ be a finite presentation of the group G. We use $F(X)$ to denote the free group on the set X, $\overline{X}$ to denote the set $X \cup X^{-1}$, and $\overline{X}^*$ to denote the free semigroup on $\overline{X}$. For $v, w \in \overline{X}^*$, we write $v =_G w$ to mean that these words represent the same element of the group presented.

We recall that if $w \in F(X)$ is a relation for $\mathcal{P}$, *i.e.* $w =_G 1$ (w is in the normal closure of the set R) then there is a van Kampen diagram for w over $\mathcal{P}$ (or a $\mathcal{P}$–diagram for w): briefly, this can be thought of as a connected, oriented, labelled, planar graph, such that all of the bounded regions, or interior faces, of its complement have boundaries labelled by words in R (read from an appropriate starting point, with an appropriate orientation), and the unbounded region has boundary label w (for more about such diagrams see for instance [17, Chap.V]). We consider here also diagrams for relations in $\overline{X}^*$.

An *area function* for $\mathcal{P}$ is a function $f : \mathbb{N} \to \mathbb{R}$ such that for each relation $w \in \overline{X}^*$ of length at most n, there is a collection of $N \le f(n)$ words $p_i \in F(X)$ and choices $r_i \in R \cup R^{-1}$, such that $w = \prod_{i=1}^{N} p_i r_i p_i^{-1}$ in the free group $F(X)$. The number N is the number of interior faces of the associated van Kampen diagram. A *radius function* for $\mathcal{P}$ is a function $g : \mathbb{N} \to \mathbb{R}$ such that for each relation $w \in F(X)$ of length at most n, there is a *not necessarily* reduced (in the sense of [17, Chap.V]) van Kampen diagram $D_{\mathcal{P}}(w)$ such that from each vertex there is a path in the 1–skeleton to the boundary $\partial D_{\mathcal{P}}(w)$ of length at most $g(n)$. Notice that reduction of a diagram reduces the area, but may in general increase the radius; there is an obvious diagram of area N and of radius zero for the unreduced word $\prod_{i=1}^{N} p_i r_i p_i^{-1}$ (this is the so-called lollypop construction, which is illustrated in [17, Chap.V] p. 237 Fig. 1.1).

An *area–radius pair*, or AR pair, for $\mathcal{P}$, is a pair (f, g) of functions $f, g : \mathbb{N} \to \mathbb{R}$ such that for each relation $w \in F(X)$ of length at most n, there is a not necessarily reduced diagram $D_{\mathcal{P}}(w)$ whose area is bounded by $f(n)$ and whose radius is bounded by $g(n)$, *i.e.* $\mathrm{Area}(D_{\mathcal{P}}(w)) \leq f(n)$ and $\mathrm{radius}(D_{\mathcal{P}}(w)) \leq g(n)$.

Our main theorem is :

THEOREM B.

Let G be a split extension of a finitely presented group K by a finitely generated free group F, so one has the short exact sequence

$$1 \to K \to G \to F \to 1.$$

If (f, g) is an AR pair for G with $f(n) \geq n$ for all n, then there is a constant $A > 1$ such that $A^g f$ is an isoperimetric inequality for K.

We define two equivalence relations on functions $\mathbb{N} \to \mathbb{R}$; we say that $f \simeq \overline{f}$ when there are integer constants A, B, C, D such that, for all $n \in \mathbb{N}$, both $f(n) \leq A\overline{f}(Bn) + Cn + D$ and $\overline{f}(n) \leq Af(Bn) + Cn + D$. Notice that all constant functions are equivalent to a linear function in this relation.

Also we have an equivalence relation $g \cong \overline{g}$ if there are integer constants A, B, C such that, for all $n \in \mathbb{N}$, $g(n) \leq A\overline{g}(Bn) + C$ and $\overline{g}(n) \leq Ag(Bn) + C$. Notice that the zero function is here equivalent to any constant function, but is *not* equivalent to a non–constant linear function. This finer equivalence relation is required in order to consider the radius function for hyperbolic groups.

The $\simeq$–equivalence class of the area function for $\mathcal{P}$ is an invariant of the group called an isoperimetric function (see for instance [1], [2], [12], [6]). The equivalence of the radius functions is more subtle, requiring *a priori* the consideration of non–reduced diagrams, and gives an invariance of the $\cong$–equivalence class of the radius functions.

Notice that if (f, g) is an AR pair for the finite presentation $\mathcal{P}$, then:

(i) f is an isoperimetric inequality for $\mathcal{P}$, and g is a radius function for $\mathcal{P}$, but they may well not be simultaneously best possible;

(ii) (f, g) is also an AR pair for the unreduced words.

The equivalence class of an AR pair for $\mathcal{P}$ is an invariant of the group so presented, in the following sense:

PROPOSITION 2.1. *Let $\mathcal{P}, \mathcal{Q}$ be finite presentations for the group G.*

If (f, g) is an AR pair for $\mathcal{P}$, then there is an AR pair $(\tilde{f}, \tilde{g})$ for $\mathcal{Q}$, such that $f \simeq \tilde{f}$ and $g \cong \tilde{g}$.

Proof. Let $\mathcal{P} = \langle X \mid R\rangle$, $\mathcal{Q} = \langle Y \mid S\rangle$, and let $w \in F(Y)$ be a relation. Rewrite each generator $y_i \in Y$ as a word $v_i(X) \in F(X)$, such that $y_i =_G v_i$. Replacing each letter y_i in w by the corresponding word v_i, gives a not necessarily reduced word $w' \in \overline{X}^*$. Let β_1 be the maximum of the lengths of the words v_i. Let $D = D_{\mathcal{P}}(w')$ be a not necessarily reduced diagram for this relation satisfying the AR pair (f, g), *i.e.* $\mathrm{Area}(D_{\mathcal{P}}(w') \leq f(\ell(w'))$ and $\mathrm{radius}(D_{\mathcal{P}}(w')) \leq g(\ell(w'))$.

Now translate this $\mathcal{P}$–diagram into a diagram over $\mathcal{Q}$ as follows. For each $x_j \in X$, there is a word $z_j \in F(Y)$ such that $x_j =_G z_j$ in G. Let β_2 be the maximum of the lengths of these words z_j. After subdivision as necessary, replace each edge of $D_{\mathcal{P}}(w')$ labelled x_j by edges labelled by the word z_j, to give a planar labelled graph D'. Each interior region of $\overline{\mathbb{R}^2 - D'}$ which in D was labelled $r_k \in R$, is now labelled by a not necessarily reduced word ${r_k}' \in \overline{Y}^*$. The unbounded region of the plane is labelled by a word $w'' \in \overline{Y}^*$ which is equal to w in G.

For each $r_k \in R$, choose a van Kampen diagram $D_{\mathcal{Q}}({r_k}')$ for ${r_k}'$, whose interior faces are labelled in S. Let α_1 be the maximum of the areas of these $\mathcal{Q}$–diagrams, and let ρ_1 be the maximum of their radii. Inserting these diagrams into the corresponding regions of D' gives a not necessarily reduced van Kampen diagram D'' over the presentation $\mathcal{Q}$ for the relation w''. The radius of D'' is at most $\rho_1 + \beta_2\mathrm{radius}(D)$.

In this procedure, each Y–letter y_i appearing in the original word w is first replaced by an X–word v_i, which is then rewritten as a word Y_i in $\overline{Y}^*$. As before, for each $y_i \in Y$ choose a $\mathcal{Q}$–diagram for the relation $Y_i =_G y_i$; let α_2 be the maximum of the areas of these diagrams, and let ρ_2 be the maximum of their radii. Adding these diagrams to the boundary of D'' gives a not necessarily reduced diagram $\overline{D}_{\mathcal{Q}}(w)$ for the original word w.

The $\mathcal{Q}$–diagram $\overline{D}_{\mathcal{Q}}(w)$ now satisfies:

$$\begin{aligned}\mathrm{Area}(\overline{D}_{\mathcal{Q}}(w)) &\leq \alpha_1\mathrm{Area}(D_{\mathcal{P}}(w')) + \alpha_2\ell(w)\\ &\leq \alpha_1 f(\ell(w')) + \alpha_2\ell(w)\\ &\leq \alpha_1 f(\beta_1\ell(w)) + \alpha_2\ell(w)\end{aligned}$$

and

$$\begin{aligned}\mathrm{radius}(\overline{D}_{\mathcal{Q}}(w)) &\leq \rho_1 + \beta_2\mathrm{radius}(D_{\mathcal{P}}(w')) + \rho_2 + \beta_1\beta_2\\ &\leq \beta_2 g(\ell(w')) + \rho_1 + \rho_2 + \beta_1\beta_2\\ &\leq \beta_2 g(\beta_1\ell(w)) + \rho_1 + \rho_2 + \beta_1\beta_2\ .\end{aligned}$$

The result now follows, using the two equivalence relations.

It is clear that for any finitely presented group satisfying an isoperimetric function f, there is an AR pair $(f, \rho f)$, where ρ is one half of the maximum of the lengths of the relations.

It has been known for a long time that hyperbolic groups have an AR pair with logarithmic radius function (a proof appeared in [7]). Since this fact is central to our proof of Theorem A (1), we include a proof of a strong form below.

LEMMA 2.2. *Let $\mathcal{P} = \langle X; R\rangle$ be a finite presentation of a group G. If G is hyperbolic, then there are constants $A, B > 0$ such that, for any relation $w \in \overline{X}^*$ with $\ell(w) \geq 1$, there is a $\mathcal{P}$–diagram of area at most $A\ell(w)(\log_2(\ell(w)) + 1)$ and of radius at most $B(\log_2(\ell(w)) + 1)$.*

Proof. We begin by recalling the definition of word hyperbolicity in terms of δ-fineness of finite presentations [1, 1.5]. We say that $\mathcal{P}$ has δ-fine geodesic triangles if for every geodesic triangle Δ in the Cayley graph Γ the fibres of the tripod map $\Delta \to T$ have diameter at most δ. [2]

Consider a relation $w = a_1 \dots a_n \in \overline{X}^*$ in G. Draw a circle in the plane, and subdivide into n vertices labelled by integers $i = 0, 1, \dots, n-1$, which we consider as representatives for their equivalence classes $\bmod n$. Map this circle to a loop in the Cayley graph Γ_X based at the identity vertex via the word w (i.e. a_i labels an edge between vertex $i-1$ and vertex $i \bmod n$). Join the vertices 0 and $[n/2]$ [3] by a straight line, and label this by a choice of a geodesic γ_1 joining the appropriate vertices in Γ_X.

For each integer $j = 2, \dots, [\log_2(n) + 1]$, and for each $i = 1, \dots, 2^j$, choose geodesics in Γ_X, the *level* j geodesics, to label the straight lines joining the vertices $[(i-1)n/2^j]$ and $[in/2^j]$ (many of these geodesics may degenerate to a point for the last j). The level j triangles are then the geodesic triangles T_j^k, for $k = 1, \dots 2^{j-1}$, with vertices $[2(i-1)n/2^j], [(2i-1)n/2^j]$ and $[2in/2^j]$, and sides consisting of two level j geodesics $\gamma_j^{2i-1}, \gamma_j^{2i}$ and a level $j-1$ geodesic γ_{j-1}^i. At the final level take the edges in the loop w for the geodesics; at this level some of the triangles may degenerate. Notice that for each j, the sum of the lengths of the level j geodesics is at most n.

As geodesic triangles are δ-fine in Γ_X, in each T_j^k the level $j-1$ edge is at distance (in Γ_X) at most δ from one of the level j edges. In fact each such triangle T_j^k can be decomposed into subregions: three triangles, many rectangles, and one central region such that:

[2] Briefly, one constructs the comparison triangle $\Delta' = ABC$ for Δ in the Euclidean plane (so Δ' is a Euclidean triangle with the same side lengths as Δ), and one considers the points a, b, c where the inscribed circle meets the sides of Δ'; here $a \in BC$, $b \in CA$, and $c \in AB$. Then $|Ab| = |Ac|$, $|Ba| = |Bc|$, and $|Ca| = |Cb|$, so one may identify Ab with Ac, Ba with Bc, and Ca with Cb to form the tripod graph T. The tripod map is the composition $\Delta \to \Delta' \to T$.

[3] $[r]$ denotes integer part of the real number r.

• the vertices of the subregions map to vertices of the Cayley graph;
• each rectangle has circumference at most $2\delta + 2$ and an edge on each of two of the sides of T_j^k;
• the central region has 6 (possibly degenerate) sides, and circumference at most $3\delta + 3$.

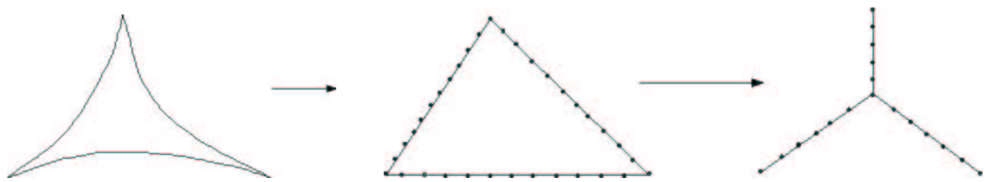

Figure 1. Fibres of the tripod map have at most 3 points, and diameter at most δ.

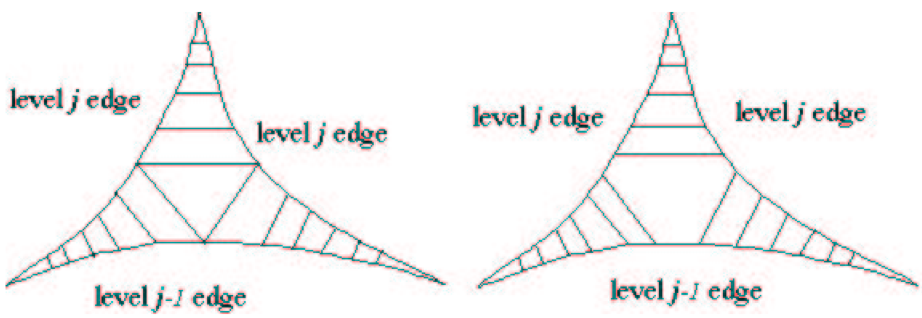

Figure 2. Two possibilities for non–degenerate triangles, drawn with straight lines joining points corresponding to vertices in the Cayley graph which are fibres of the tripod map; each has length at most δ (in Γ_X). In the first case, suppress one edge of the central triangle.

If the central region is a geodesic triangle, as in one of the figures, then remove one of the sides, to give a pentagon, with one side on a level $j-1$ edge, one on a level j edge. The central region of the triangle Δ is bounded by a (possibly degenerate) geodesic hexagon, with 2 or 3 edges of length 1 on the 3 sides of Δ together with 3 sides of length at most δ. Apart from this central region, there are 3 geodesic triangles each having two long sides within δ-neighbourhoods of each other and one side of length at most δ. It is clear that these latter triangles can be decomposed into rectangles, each with two opposite sides of length (at most) 1 and two opposite sides of length at most δ.

Filling in each rectangle and central region of T_j^k with a $\mathcal{P}$–diagram gives a $\mathcal{P}$–diagram D for w. In the 1–skeleton of this diagram there is a path from the boundary of a triangle T_j^k to the outside boundary by passing always to a higher level side along a path of length at most δ. Also at level j there at most $[n/2^{j-1}]+1$ triangles, and the level j part of their boundaries has total length at most n . Suppose that all diagrams of circumference at most $3\delta + 3$ have radius at most C_1 and area at most C_2. It is clear that D has radius at most $C_1 + \delta([\log_2(n)] + 1) \leq$

$B(\log_2(\ell(w))+1)$. As each of the (rectangular or central) regions within each level j triangle has at least two edges on the union of the level j sides and the level $j-1$ sides, there at most n such regions at each level. It follows that the area of D is at most $nC_2([\log_2(n)]+1) \leq A\ell(w)(\log_2(\ell(w))+1)$.

In the same way, it is not hard to see from the proofs in [8, pages 52 and 152] that if G is a synchronously (respectively asynchronously) automatic group then there are constants $A, B > 0$ (resp. $C > 1,\ D > 0$) such that (Ax^2, Bx) (resp. (C^x, Dx)) is an AR pair for G.

Remark. All minimal area van Kampen diagrams in a finite presentation of a hyperbolic group have a uniform logarithmic upper bound on their radii. This is asserted in [14] p. 100, and a proof is given in [16] Corollary 4.2.

3. New diagram from an automorphism

Let $\mathcal{P} = \langle X \mid R \rangle$ be a finite presentation of the group G, and let $\phi : G \to G$ be an automorphism.

For each $x_j \in X$, choosing a word in $F(X)$ representing $\phi(x_j)$ induces a semigroup homomorphism $\Phi : \overline{X}^* \to \overline{X}^*$ such that $\Phi(x_j) =_G \phi(x_j)$, and $\Phi(x_j^{-1}) =_G \phi(x_j^{-1}) =_G (\phi(x_j))^{-1}$. As ϕ is an *auto*morphism, there is also a semigroup homomorphism $\Psi : \overline{X}^* \to \overline{X}^*$ such that $\Psi(\Phi(x_j^{\pm 1})) =_G x_j^{\pm 1} =_G \Phi(\Psi(x_j^{\pm 1}))$. Rapaport's theorem, used in [9], says that there is a finite set of generators Z for G, such that Φ can be chosen to be an automorphism of $F(Z)$. We shall *not* make use of Rapaport's theorem in this note.

We make a choice once and for all of a $\mathcal{P}$-diagram $D_{\mathcal{P}}(w)$ for each relation w. It is a straightforward matter to obtain a not necessarily reduced diagram $D'_{\mathcal{P}}(\Phi(w))$ for the not necessarily reduced word $\Phi(w)$, as follows:

(i) Subdivide and relabel the 1–skeleton of $D_{\mathcal{P}}(w)$ so that an edge previously labelled x_i is subdivided and relabelled $\Phi(x_i)$.

This gives a connected, planar, labelled, oriented 1–complex D' in the plane, whose outside boundary is labelled by the not necessarily reduced word $\Phi(w)$. The compact regions are labelled by words $\Phi(r)$ where $r \in R$. Each $\Phi(r)$ is a relation in G; choose a diagram $D_{\mathcal{P}}(\Phi(r))$ for each $\Phi(r)$.

(ii) Insert the diagrams $D_{\mathcal{P}}(\Phi(r))$ into the corresponding faces of the planar graph D' to obtain $D'_{\mathcal{P}}(\Phi(w))$.

We have thus shown that:

LEMMA 3.1.

There is a positive constant S such that if $D_{\mathcal{P}}(w)$ is a $\mathcal{P}$–diagram for the relation w, then there is a $\mathcal{P}$–diagram $D'_{\mathcal{P}}(\Phi(w))$ such that $\mathrm{Area}(D'_{\mathcal{P}}(\Phi(w))) \leq S\mathrm{Area}(D_{\mathcal{P}}(w))$.

It suffices to take S to be the maximum of the areas of the diagrams $D_{\mathcal{P}}(\Phi(r))$, $r \in R$.

The "inverse" map on diagrams is slightly more complicated:

LEMMA 3.2. *There are positive constants S', S'' such that if $D'_{\mathcal{P}}(\Phi(w))$ is a diagram for the relation $\Phi(w)$, then there is a diagram $D'''_{\mathcal{P}}(w)$ for the relation w such that* $\mathrm{Area}(D'''_{\mathcal{P}}(w)) \leq S'\mathrm{Area}(D'_{\mathcal{P}}(\Phi(w))) + S''\ell(w)$.

Proof. As before there is a diagram $D''_{\mathcal{P}}(\Psi(\Phi(w)))$ for the relation $\Psi(\Phi(w))$, whose area is at most S' times the area of the diagram $D'_{\mathcal{P}}(\Phi(w))$, where S' is the maximum of the areas of the diagrams $D_{\mathcal{P}}(\Psi(r))$.

If $w = a_1 a_2 \ldots$, where $a_i \in \overline{X}$, then $\Psi(\Phi(w)) = \Psi(\Phi(a_1))\Psi(\Phi(a_2))\ldots$ in $\overline{X}^*$. For each $x_i \in X$, choose a diagram for the $\mathcal{P}$ relation $x_i =_G \Psi(\Phi(x_i))$; let S'' be the maximum area of these diagrams. Adding these to the boundary of $D''_{\mathcal{P}}(\Psi(\Phi(w)))$ gives a not necessarily reduced diagram for w over $\mathcal{P}$ with the required area.

4. Proof of the theorem B

Let $\mathcal{P}_K = \langle x_1, \ldots, x_m \mid r_1, \ldots, r_k \rangle$ be a finite presentation of the group K, and let F_n be the free group of rank n, generated by elements $\{t_1, \ldots, t_n\}$. For a split extension $1 \to K \to G \to F_n \to 1$, and for each t_i, there is an automorphism $\phi_i : K \to K$ given by conjugation by t_i restricted to K. Let Φ_i be a lift of ϕ_i to a semigroup endomorphism and let Ψ_i be a lift of the inverse, as in the preceding section. Then G has a presentation

$$\mathcal{P}_G = \langle x_1, \ldots, x_m, t_1, \ldots, t_n \mid r_1, \ldots r_k, \{t_i^{-1} x_j t_i \Phi_i(x_j)^{-1}\} \rangle \quad .$$

In a diagram over $\mathcal{P}_G$ for a relation w which does not involve any of the stable letters t_i for the HNN-extension G, the regions corresponding to relations $t_i^{-1} x_j t_i \Phi_i(x_j)^{-1}$ form "t–rings" (as in [9, §5]); each t–ring is an annular region and is associated to one of the stable letters.

Let $w \in \overline{X}^*$ be a relation over $\mathcal{P}_G$, and let $D = D_{\mathcal{P}}(w)$ be a $\mathcal{P}_G$–diagram for w. Let A be a t–ring, and let D_0, D_1 be the two components of $\overline{D - A}$, where D_1 is the outer component, containing the boundary of D, and D_0 is the inner component. Let u_0 be the label on the boundary of D_0 (the inner boundary of A), and let u_1 be the label on the outer boundary component of A; each is a word in $\overline{X}^*$ which is a relation in $\mathcal{P}_G$ (and in $\mathcal{P}_K$). The subdiagram D_0 is a diagram for u_0 over $\mathcal{P}_G$.

In a $\mathcal{P}_G$–diagram for w, the t–rings are nested at most as deep as the radius of the diagram. Consider an innermost t–ring A in D, where the t–edges in A are labelled t_i; in this case, D_0 is a $\mathcal{P}_K$–diagram for u_0.

There are two cases to consider according to the orientation the edge t_i in A: either $u_1 = \Phi_i(u_0)$, or $u_0 = \Phi_i(u_1)$.

In case 1, applying Φ_i to the P_K–diagram D_0, Lemma 3.1 says that there is a $\mathcal{P}_K$–diagram $D(u_1)$ for u_1 of area at most $S.\mathrm{Area}(D_0)$. Thus replacing $A \cup D_0$ in D by $D(u_1)$, and simultaneously doing the analogous procedure on all such innermost t–rings, at worst multiplies the area of D by S.

In case 2, Lemma3.2 shows how to obtain a $\mathcal{P}_K$–diagram $D(u_1)$ of area at most $S'\mathrm{Area}(D_0) + S''\ell(u_1)$. Each edge in D occurs in at most two t–rings, so twice the number of edges in D gives a bound on $\ell(u_1)$; thus $\ell(u_1) \leq \ell(w) + 2\rho\ \mathrm{Area}(D)$, where ρ is the maximum length of a relation in $R \cup \{t_i x_j {t_i}^{-1} \Phi_i(x_j)^{-1}\}$. Thus replacing $D_0 \cup A$ in D by $D(u_1)$, and doing this on all innermost t–rings simultaneously, at worst multiplies the area by S', and adds on $S''(\ell(w) + 2\rho\ \mathrm{Area}(D)) \leq Mf(\ell(w))$, for some positive constant M, independent of w (this makes use of the assumption that $f(n) \geq n$ for all n). Choose $M \geq \max(S, S', 1)$.

These changes must be performed at most $g(\ell)$ times (where we let $\ell = \ell(w)$), as the nesting of t–rings is bounded by the radius. Then removing all t–rings gives a $\mathcal{P}_K$–diagram for w of area at most

$$\underbrace{M(\ldots(M(Mf(\ell) + Mf(\ell)) + Mf(\ell)) + \ldots + Mf(\ell))}_{\text{at most } g(\ell) \text{ times}} \leq M^{g(\ell)+1} f(\ell).$$

Theorem B follows by replacing M by an even larger constant $A > 1$ ($A > M^2$ will work).

Theorem A now follows from applying Theorem B to the AR pair in Lemma 2.2.

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