BOUNDED COCYCLES AND COMBINGS OF GROUPS

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Abstract. We adopt the notion of combability of groups defined in [Ghys2]. An example is given of a (bi-)comparable group which is not residually finite. Two of the eight 3-dimensional geometries, $\mathbb{S}^2_1(\mathbb{R})$ and $\mathbb{H}^2 \times \mathbb{R}$, are quasiisometric. Seifert fibred manifolds over hyperbolic orbifolds have bicomparable fundamental groups. Every combable group satisfies an exponential isoperimetric inequality.

§0. Introduction

In this article we shall give applications of bounded cohomology to combinatorial group theory. We shall produce an example of a combable group which is not residually finite. The example is due to Raghunathan [Ra1] and has been extensively studied by Toledo [To]. This relates to a question raised in [G1], whether automatic groups are residually finite, since all automatic groups are combable.

Before describing more of our results, we recall the definition of a combing of a group [Ghys2]. If $G$ is a finitely generated group with finite set $A$ of semigroup generators, we consider the Cayley graph $\Gamma$ for these generators as a metric space equipped with the word metric. The free monoid $A^*$ on $A$ can be considered as paths in $\Gamma$ starting at the base point 1 (identifying the vertices of $\Gamma$ with $G$), parametrized by arc length, and ending at the group element $w \in G$ represented by the word $w \in A^*$; such a path may then be considered as extended to all positive reals by remaining constant at the end point for all times greater than the length of the word $w$. A combing is a section $\sigma : G \to A^*$ of the end point mapping $A^* \to G$ such that the $k$-fellow traveller is satisfied for some $k \geq 0$; this means that

$$|\sigma(ga)(t) - \sigma(g)(t)| \leq k,$$

for all $g \in G$, $a \in A$, $t \geq 0$, where we have denoted the distance between two points $x, y \in \Gamma$ by $|x - y|$, following the convention of [Gr2]. It is known that combability of a group is a quasiisometry invariant property [AB][Sh], so, in particular, does not depend on the particular finite set $A$ of generators chosen.

The notion of combability we have adopted is due to Thurston. We first heard it in his lecture at MSRI in January 1989. It first appeared in print in [Ghys2] and it was developed in [AB] and [Sh]. A word of caution is necessary, since the notion of

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Combability adopted in [ECH] is a different and stronger notion (combable groups in the sense of [ECH] have linearly bounded combings, in the sense of Section 3 below); this has led to some confusion in comparing statements of results. Combable groups are a natural generalization of automatic groups [ECH] (precise definitions are given in Section 3 below) in which, roughly speaking, one drops the regular language but preserves the geometry. To what extent they properly generalize automatic groups is a moot question at the moment of writing.

A tool in our results is an Integrality Theorem (Theorem 1.3 below), which relates bounded integral cohomology to bounded real cohomology. It says that if an integral group cohomology class is bounded when it is considered as a real cohomology class, then it is bounded as an integral cohomology class. Although the result is easy to prove, we have found no prior reference to it in the literature.

We also prove that uniform lattices in $SL_2(\mathbb{R})$ are quasiisometric to uniform lattices in $SO^+(2,1) \times \mathbb{R}$. As a consequence, the only coincidence among the 8 geometries of 3-manifolds up to quasiisometry is between $SL_2(\mathbb{R})$ and $\mathbb{H}^2 \times \mathbb{R}$.

In Section 4, we refine our results on combings to produce bicombings on certain central extensions (the definition of bicombing appears in Section 3 below). Bicomblings were introduced by H. Short in [Sh] as a generalization of bi-automatic groups. They are more difficult to deal with since, unlike combability, which is a quasiisometry invariant property [AB][Sh], it is not known whether the existence of a bicombing is a quasiisometry invariant condition. However bicombinability has implications about the structure of a group that do not follow just from the existence of a combing. In Theorem 4.1, we prove that if $E$ is a central extension of $\mathbb{Z}$ by a bicomable group $G$ which is defined by a bounded cocycle, then $E$ is bicomable. It follows that Raghunathan’s group is bicomable. We deduce furthermore in Theorem 4.4 that every Seifert fibred 3-manifold over a hyperbolic 2-dimensional orbifold has a bicomable fundamental group.

In Section 5, we prove an exponential isoperimetric inequality for combable groups. It is shown in Proposition 3.9 that a group possessing a linearly bounded combing satisfies a quadratic isoperimetric inequality; this is the situation in combings arising from automatic structures and from combings arising from convex metrics in differential geometry, but our definition of combing (which is the same as that of [Ghys2] and [Sh]) does not require linear boundedness (unlike the definition adopted by [ECH]).

Domingo Toledo has written an Appendix containing a proof that the characteristic class of Raghunathan’s central extension is bounded. The group of this extension is an example of a (bi-)combable group which is not residually finite. It is a pleasure to thank him for his suggestions. We are grateful to John Stallings for his suggestion that our “Integrality Theorem 1.3” should be deduced from the Mayer-Vietoris exact sequence. There is some overlap between this paper and [HP], which was written between the appearance of a preliminary draft of this paper and the final version. We thank the referee for a simplification in the proof of Theorem 3.1. Finally we thank M. Bridson for helping us to assign credit for ideas in an area where almost nothing is yet in print.
§1. Bounded cohomology

In this section $G$ will denote a discrete group and $A$ will denote either $\mathbb{Z}$, $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$, considered as a $G$-module with trivial action. We recall [Se] that the ordinary group cohomology $H^*(G, A)$ is given as the cohomology of the cochain complex $C^*(G, A)$, where $C^n(G, A)$ consists of mappings $G^n \to A$ ($G^n$ is the $n$-fold cartesian product of $G$, $n \geq 0$) and where the differential $\delta : C^n(G, A) \to C^{n+1}(G, A)$ is given by the formula

$$\delta f(g_1, g_2, \ldots, g_{n+1}) = f(g_2, g_3, \ldots, g_{n+1}) - f(g_1 g_2, g_3, \ldots, g_{n+1}) + \cdots + (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + \cdots + (-1)^{n+1}f(g_1, g_2, \ldots, g_n).$$

This is a cohomological functor and corresponding to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$ is the long exact sequence of cohomology groups.

If now $A = \mathbb{Z}$ or $\mathbb{R}$, we consider cochains $f \in C^n(G, A)$ for which there exists $M_f > 0$ such that $|f(g_1, g_2, \ldots, g_n)| \leq M_f$ for all $g_1, g_2, \ldots, g_n \in G$. The set $C^*_b(G, A)$ of such cochains is a subgroup and a simple calculation shows that $\delta f \in C^{n+1}_b(G, A)$ if $f \in C^n_b(G, A)$, so we have the cochain complex $C^*_b(G, A)$ of bounded cochains with values in $A$. Thus we can define $H^*_b(G, A) = H^n(C^*_b(G, A))$. The inclusion homomorphism $C^*_b(G, A) \to C^*(G, A)$ induces homomorphisms $H^*_b(G, A) \to H^*(G, A)$. One says a class $z \in H^n(G, A)$ is bounded if it is the image of this homomorphism.

**Proposition 1.1.** There is a commutative diagram whose rows are long exact sequences of bounded cohomology groups and ordinary cohomology groups

$$
\begin{array}{cccccc}
H^{n-1}(G, \mathbb{R}/\mathbb{Z}) & \longrightarrow & H^*_b(G, \mathbb{Z}) & \longrightarrow & H^*_b(G, \mathbb{R}) & \longrightarrow & H^n(G, \mathbb{R}/\mathbb{Z}) \\
\downarrow \cong & & \downarrow & & \downarrow & & \downarrow \cong \\
H^{n-1}(G, \mathbb{R}/\mathbb{Z}) & \longrightarrow & H^n(G, \mathbb{Z}) & \longrightarrow & H^n(G, \mathbb{R}) & \longrightarrow & H^n(G, \mathbb{R}/\mathbb{Z}) \\
\end{array}
$$

**Proof.** We must show that for each $n$ there is a short exact sequence

$$0 \to C^n_b(G, \mathbb{Z}) \to C^n_b(G, \mathbb{R}) \to C^n(G, \mathbb{R}/\mathbb{Z}) \to 0.$$ 

The only point that is not completely trivial is that the natural homomorphism $C^n_b(G, \mathbb{R}) \to C^n(G, \mathbb{R}/\mathbb{Z})$ is surjective. To see this, let $f \in C^n(G, \mathbb{R}/\mathbb{Z})$. Then we can lift $f$ to a cochain $F \in C^n(G, \mathbb{R})$ taking values in $[0, 1)$. It follows that $F$ is bounded, and surjectivity follows.

Standard results of homological algebra then give the long exact sequence of bounded cohomology groups, and the commutativity of the diagram follows from the naturality of the constructions.
Corollary 1.2. One has the exact Mayer-Vietoris sequence

\[ \to H^{n-1}(G, \mathbb{R}) \to H^n_b(G, \mathbb{Z}) \to H^n(G, \mathbb{Z}) \oplus H^n_b(G, \mathbb{R}) \to H^n(G, \mathbb{R}) \to . \]

Proof. The deduction of the Mayer-Vietoris exact sequence from Proposition 1.1 follows exactly as in the singular homology theory.

Example. It is a theorem of Trauber’s [Gr1] that bounded cohomology groups of amenable groups with real coefficients vanish, so \( H^1_b(\mathbb{Z}, \mathbb{R}) = H^2_b(\mathbb{Z}, \mathbb{R}) = 0 \). Since \( \mathbb{Z} \) has cohomological dimension 1, it follows from Corollary 1.2 that

\[ H^2_b(\mathbb{Z}, \mathbb{Z}) = \frac{H^1(\mathbb{Z}, \mathbb{R})}{H^1(\mathbb{Z}, \mathbb{Z})} = \mathbb{R}/\mathbb{Z}. \]

For an interpretation of this result in terms of the rotation number of a homeomorphism of the circle, see [Ghys1].

Theorem 1.3 (Integrality Theorem). If \( P := H^2(G, \mathbb{Z}) \otimes_{H^n(G, \mathbb{R})} H^2_b(G, \mathbb{R}) \), then the canonical map \( H^2_b(G, \mathbb{Z}) \to P \) is surjective.

Proof. This is immediate from Corollary 1.2.

Example 1.4. If \( M \) is a closed orientable surface of genus \( g > 1 \) and \( G = \pi_1(M) \), then the fundamental class \( [M] \in H^2(G, \mathbb{R}) \) is bounded [Gr1]. It follows from the Integrality Theorem that \( H^2(G, \mathbb{Z}) \) is the image of \( H^2_b(G, \mathbb{Z}) \).

§2. A Certain Central Extension

In this section, let \( m \geq 3 \). Let \( \Gamma \) be a torsion-free cocompact lattice in \( \text{SO}^+(2, m) \), the connected component of the identity element in the Lie group \( \text{SO}(2, m) \). We consider the commutative diagram of central extensions

\[
\begin{array}{ccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & E & \longrightarrow & \Gamma & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widehat{\text{SO}}^+(2, m) & \longrightarrow & \text{SO}^+(2, m) & \longrightarrow & 1.
\end{array}
\]  

(2.1)

Here \( \Gamma \to \text{SO}^+(2, m) \) is the inclusion homomorphism and \( \widehat{\text{SO}}^+(2, m) \) is the \( \mathbb{Z} \)-cover of \( \text{SO}^+(2, m) \) (note that \( \text{SO}^+(2, m) \) has the maximal compact subgroup \( K = \text{SO}(2) \times \text{SO}(m) \) onto which it deformation retracts, and \( \pi_1(K) = \mathbb{Z} \times \mathbb{Z}_2 \), since \( m \geq 3 \); thus there is a unique infinite cyclic cover of \( \text{SO}^+(2, m) \)). Raghunathan’s main theorem of [Ra1] implies that every subgroup \( \Delta \) of finite index in \( E \) contains \( 8\mathbb{Z} \), where \( \mathbb{Z} \) denotes the central kernel in the top line of (2.1) (compare [To], Lemma 3.4). In particular, it follows that \( E \) is not residually finite.
Theorem 2.2. The characteristic class of the central extension which is the first row in (2.1) is represented by a bounded integral cocycle. The group $E$ there is finitely presented but not residually finite.

Proof. In the Appendix to this article due to Domingo Toledo it is proved in Proposition A3 that the characteristic class of the extension is represented by a bounded real cocycle. By the Integrality Theorem 1.3, it follows that this class is represented by a bounded integral cocycle. The last assertion of the theorem is a restatement of Raghunathan’s result. This completes the proof.

§3. The word metric on a central extension given by a bounded cocycle

In this section we consider a central extension

$$1 \rightarrow \mathbb{Z} \rightarrow E \rightarrow G \rightarrow 1,$$

whose characteristic class is defined by a 2-cocycle $f : G \times G \rightarrow \mathbb{Z}$ taking only a finite set of values in $\mathbb{Z}$. By Theorem 1.3 this happens if and only if the image of the characteristic class in $H^2(G, \mathbb{R})$ is bounded. We assume that $f$ is normalized, so that $f(g, 1) = f(1, g) = 1$ for all $g \in G$. This amounts to altering $f$ by the constant coboundary $f(1, 1)$, which does not alter the fact that the value set is finite. Here we write the central kernel isomorphic to $\mathbb{Z}$ multiplicatively, with generator $t$, say. We shall first calculate the word metric on $E$.

Let $A = A^{-1}$ be a finite set of generators for $G$ and denote by $|g|_G = d(1, g)$, the distance in the word metric of $G$. As a set, $E$ is $\langle t \rangle \times G$, while the group law is $(t^n, g)(t^m, g') = (t^{n+m} f(g, g'), gg')$. We take as semigroup generators for $E$ the set $B = \{(t^{\pm 1}, 1), (1, a) \mid a \in A\}$ and denote by $|(t^n, g)|_E$ the distance in the word metric of the element $(t^n, g)$ from the identity $(1, 1)$ (we used here the fact that $f$ is normalized). The word metric on the infinite cycle $\langle t \rangle$ for the generators $\{t^{\pm 1}\}$ is given by $|t^n| = |n|$.

If $h, h' : S \rightarrow \mathbb{R}$ are functions defined on the set $S$ taking non-negative real values, then we write $h \sim h'$ if there are positive constants $\alpha, \beta$ such that $h'(s) \leq \alpha h(s) + \beta$ and $h(s) \leq \alpha h'(s) + \beta$ for all $s \in S$. In the application that follows, $S = E$, as a set, and $h, h' : E \rightarrow \mathbb{R}$ are given by $h(t^n, g) = |(t^n, g)|_E$, $h'(t^n, g) = |n| + |g|_G$, respectively, for $n \in \mathbb{Z}$, $g \in G$; we shall abuse the notation by abbreviating these functions by their values.

Theorem 3.1. Under the above assumptions we have $|(t^n, g)|_E \sim |n| + |g|_G$ for all $n \in \mathbb{Z}$, $g \in G$. Moreover the group $E$ is quasiisometric to the product $\mathbb{Z} \times G$.

We shall deduce this result from the following two Lemmas.

Lemma 3.2. Assume that $|f(g, g')| \leq M$ for all $g, g' \in G$, with $M \geq 1$. Then $|(t^n, 1)|_E \geq \frac{1}{M} |n|$ for all $n \in \mathbb{Z}$.

Proof. Suppose we have $(t^n, 1) = y_1 y_2 \cdots y_k$, where either $y_i = (t^{\pm}, 1)$ or $y_i = (1, a)$ for $a \in A$. Let $r$ denote the number of generators $y_i$ in the product of the second
type and let \( s \) be the number of factors of the first type. By centrality, we can move all the factors of the second type to the front of the expression to get

\[
(t^n, 1) = (t^{e_1}, 1)(t^{e_2}, 1) \cdots (t^{e_s}, 1)(1, a_1)(1, a_2) \cdots (1, a_r),
\]

where \( a_i \in A \) and where \( e_i = \pm 1 \). Furthermore, using the cocycle condition in the definition of the product in \( E \), we see that

\[
(1, a_1)(1, a_2) \cdots (1, a_r) = (t^n, a_1a_2 \cdots a_r) = (t^n, 1),
\]

\(|u| \leq Mr \). In addition we see that \( s \geq |n| - |u| \), since the sum of the exponents in \( t \) must add up to \( n \). Hence we have \(|n| \leq s + |u| \leq s + Mr \leq M(s + r) = Mk \), or \( k \geq \frac{|n|}{M} \). Since this holds for every description of \((t^n, 1)\) as a product of generators, it follows that \(|(t^n, 1)|_E \geq \frac{|n|}{M} \), and the lemma is established.

**Lemma 3.3.** We have \(|(1, g)|_E \leq (M + 1)|g|_G \) and \(|(1, g)^{-1}|_E \leq M + (M + 1)|g|_G \) for all \( g \in G \).

**Proof.** Let \( r = |g|_G \) and let \( g = a_1a_2 \cdots a_r \), with \( a_i \in A \). Then we have

\[
\prod_{i=1}^{r}(1, a_i) = (t^n, g), \text{ where } |u| \leq Mr.
\]

It follows that \(|(1, g)|_E \leq Mr + r = (M + 1)|g|_G \). On the other hand, \((1, g)^{-1} = (t^n, 1)(1, g^{-1})\), where \(|u| \leq M \). It follows that \(|(1, g)^{-1}|_E \leq M + |(1, g^{-1})|_E \leq M + (M + 1)|g|_G \), where we made use of the first part of the proof and the equality \(|g^{-1}|_G = |g|_G \), which holds since \( A = A^{-1} \).

**Proof of Theorem 3.1.** Since \((t^n, g) = (t^n, 1)(1, g)\), we have by the triangle inequality and Lemma 3.3

\[
|(t^n, g)|_E \leq |n| + (M + 1)|g|_G \leq (M + 1)(|n| + |g|_G).
\]

In the opposite direction, observe that Lemma 3.2 holds for \((t^n, g)\). Then

\[
|(t^n, g)|_E \geq |n|/M \text{ and } |(t^n, g)|_E \geq |g|_G,
\]

so

\[
|t^n, g|_E \geq \frac{1}{2M}(|n| + |g|_G).
\]

The combination of (3.4) and (3.5) gives \(|(t^n, g)|_E \sim |n| + |g|_G \). The last assertion of the Theorem follows from a straightforward calculation, making use of the left invariance of the word metrics of \( E \) and \( \mathbb{Z} \times G \), the boundedness of the cocycle, and the centrality of the subgroup \( \langle (t, 1) \rangle \) of \( E \). We give the details for completeness. The quasiisometry between the two Cayley graphs will be the identity map of the set \( E = \langle t \rangle \times G \) and we use the same semigroup generators \( B = \{ (t^{\pm 1}, 1), (1, a) \mid a \in A \} \) as above for each of the two group laws. To distinguish these laws, we denote by \((t^n, g)_E(t^m, h)_E = (t^{m+n}f(g, h), gh)_E\) the product in \( E \) (so we attach the subscript \( E \) to the element to indicate the group law in \( E \)) and we denote by \((t^n, g)\_\Pi(t^m, h)_\Pi = (t^{m+n}, gh)_\Pi \) the group law in the
product group $\langle t \rangle \times G$. We observe that $(t^n, g)^{-1}_E = (t^{-n+u}, g^{-1})_E$, where $u$ satisfies $|u| \leq M$.

Now calculate

$$d_E((t^n, g)_E, (t^m, h)_E) = d_E(1, (t^n, g)\_E^{-1}(t^m, h)_E)$$

$$= |(t^{m-n+v}, g^{-1}h)|_E, \quad \text{where } |v| \leq 2M,$$

$$\leq \alpha(|m - n + v| + |g^{-1}h|_G) + \beta \quad \text{for constants } \alpha, \beta$$

$$= \alpha d_{\Pi}((t^n, g)_\Pi, (t^m, h)_\Pi) + \beta', \quad \text{with } \beta' \text{ constant.}$$

On the other hand, we have

$$d_{\Pi}((t^n, g)_\Pi, (t^m, h)_\Pi) = |m - n| + |g^{-1}h|_G$$

$$\leq \alpha d_E(1, (t^{m-n}, g^{-1}h)_E) + \beta \quad \text{with constants } \alpha, \beta$$

$$= \alpha d_E((t^n, g)_E, (t^n, g)_E(t^{m-n}, g^{-1}h)_E) + \beta$$

$$\leq \alpha d_E((t^n, g)_E, (t^{m+n}, h)_E) + \beta, \quad \text{with } |u| \leq M,$$

$$\leq \alpha d_E((t^n, g)_E, (t^m, h)_E) + d_E((t^m, h)_E, (t^m, h)_E(t^n, 1)_E) + \beta$$

$$\leq \alpha d_E((t^n, g)_E, (t^m, h)_E) + \alpha M + \beta.$$

The last two paragraphs complete the verification that the group $E$ is quasiisometric to the product $\mathbb{Z} \times G$, and the proof of Theorem 3.1 is complete.

**Definition.** Let $G$ be a finitely generated group with finite set $A$ of semigroup generators. Let $X$ denote the Cayley graph, equipped with the word metric. Thus the set of vertices of $X$ is $G$ and the set of edges is the set of all triples $(g, a, g')$, where $g, g' \in G$, $a \in A$, and $g' = ga$. Denote the word metric by $d$. Then $G$ acts on $X$ on the left by isometries. Let $A^*$ be the free monoid on $A$ and let the evaluation mapping $A^* \to G$ be denoted $w \mapsto \tilde{w}$. A word $w \in A^*$ is considered as a path $w(t)$ in $X$ defined for all $t \geq 0$, starting at $1 \in G$, parametrized by arc length for $0 \leq t \leq \ell(w)$, where $\ell(w)$ is the length of $w$, and remaining constant at $\tilde{w}$ for all $t \geq \ell(w)$.

A combing of $G$ is a section $\sigma : G \to A^*$ of the evaluation mapping $A^* \to G$ which possesses a *modulus of continuity* $k > 0$. This means for all $g \in G$, $a \in A$ and all $t \geq 0$ we have $d(\sigma(g)(t), \sigma(ga)(t)) \leq k$.

The combing $\sigma$ of $G$ is said to be a *bicombing* [Sh] if, in addition, for all $g \in G$, $a \in A$, $t \geq 0$ one has

$$d(a \cdot \sigma(g)(t), \sigma(ag)(t)) \leq k.$$
bicomable if, for some finite set of genertators, it admits a bicombing. It is an open question whether all combable groups are bicomable.

Examples of combable groups are the fundamental groups of all compact convex Riemannian manifolds of nonpositive sectional curvature. This is a reflection of the convexity of the metric.

The group $G$ is automatic if it is combable and if the section $\sigma$ above is such that $\sigma(G)$ is a regular sublanguage of $A^*$. This definition is known to be equivalent to that given in [ECH]. The automatic group is said to be biautomatic if the underlying combing $\sigma$ possesses a modulus of bicontinuity.

All word hyperbolic groups, in the sense of Gromov [Gr2], are automatic and hence combable. All compact 3-manifolds satisfying Thurston’s geometrization conjecture and possessing no Nil nor Sol pieces in their toral decomposition have automatic fundamental groups [ECH]. It is an open question whether the fundamental group of a compact convex Riemannian manifold of nonpositive curvature is automatic.

**Definition.** Let $G$ be a finitely generated group and $A$ a finite set of semigroup generators. We say a section $\sigma : G \to A^*$ of the evaluation map $A^* \to G$ is linearly bounded if there are constants $\alpha, \beta$ so that
text{for all $g \in G$. A combing (resp. bicombing) is linearly bounded if it is so when considered as a section of the evaluation mapping.}

**Example.** If $M$ is a compact convex Riemannian manifold of nonpositive curvature, the argument of [GH], Proposition 19, when combined with the comparison theorems of differential geometry, shows that $\pi_1(M)$ is combable with a combing which is linearly bounded (in fact, the combing consists of quasigeodesics for the word metric).

**Corollary 3.6.** Suppose we have a central extension
\begin{equation}
1 \to \mathbb{Z} \to E \to G \to 1,
\end{equation}
where $G$ is combable. Suppose that the characteristic class of the extension in $H^2(G, \mathbb{Z})$ is represented by a 2-cocycle which takes only a finite set of values in $\mathbb{Z}$. Then $E$ is combable. In addition, if $G$ is combable with a linearly bounded combing, then $E$ is combable with a linearly bounded combing.

**Proof.** We have $E$ is quasiisometric to $\mathbb{Z} \times G$, by Theorem 3.1. Since $G$ is combable, it follows that $\mathbb{Z} \times G$ is also combable [Sh]. Since the existence of a combing is a quasiisometry invariant [Sh], it follows that $E$ is combable.

\footnote{This follows in effect because $\mathbb{Z}$ has a “short combing”, in the terminology of [Sh], in view of the observation that if $G$ and $H$ are combable groups where $G$ admits a short combing, then $G \times H$ is combable (with a short combing if $H$ also has a short combing). We remind the reader that a combing $\sigma : G \to A^*$, with $A$ a finite set of semigroup generators for $G$, is short if there is a constant $M$ so that $|\ell(\sigma(ga)) - \ell(\sigma(g))| \leq M$ for all $g \in G$ and $a \in A$. In general, if $G$ and $H$ are combable, then it is an open question whether $G \times H$ is combable, unless one makes additional assumptions on the groups.}
A calculation shows that the product combing on \( \mathbb{Z} \times G \) is linearly bounded if that on \( G \) is linearly bounded. It follows from examining the construction of the combing on the quasiisometric group \( E \) [Sh, 1.2] that this combing is also linearly bounded.

**Corollary 3.7.** Raghunathan’s group \( E \) in (2.1) is combable but not residually finite, for \( m \geq 3 \).

This is immediate from Theorem 2.2, Corollary 3.6 and the example immediately preceding it.

**Remark.** This result applies in particular to Raghunathan’s central extension group \( E \) in (2.1), so \( E \) is combable with a linearly bounded combing. We shall show in the next section that \( E \) is in fact bicomiable with a linearly bounded bicombing. It is an open question whether \( E \) is (bi-)automatic, and, indeed, it is unknown whether the uniform lattice \( \Gamma \) in (2.1) is (bi-)automatic.

**Definition.** If \( \Gamma_1 \) and \( \Gamma_2 \) are finitely generated groups, we write \( \Gamma_1 \sim \Gamma_2 \) if their Cayley graphs for some (and hence for any) finite set of generators are quasiisometric [GH].

**Corollary 3.8.** If \( \Gamma_1, \Gamma_2 \) are uniform lattices in the Lie groups \( \tilde{\text{SL}}_2(\mathbb{R}), \text{SO}^+(2,1) \times \mathbb{R} \) respectively, then \( \Gamma_1 \sim \Gamma_2 \).

**Proof.** Let \( M \) be a closed orientable hyperbolic surface and let \( T \) be the unit tangent circle bundle to \( M \). Then \( G_1 := \pi_1(T) \) is a uniform lattice in \( \tilde{\text{SL}}_2(\mathbb{R}) \) [Sc] and \( G_2 := \pi_1(M) \times \mathbb{Z} \) is a uniform lattice in \( \text{SO}^+(2,1) \times \mathbb{R} \). However both \( G_1 \) and \( G_2 \) are central extensions with kernel \( \mathbb{Z} \) of \( \pi_1(M) \) and both are defined by bounded cocycles. It follows that \( \Gamma_1 \sim \Gamma_2 \). However \( \Gamma_i \sim G_i \) for \( i = 1, 2 \) by [GH], Proposition 19, so we get \( \Gamma_1 \sim \Gamma_2 \).

**Remark.** It follows from Corollary 3.8 that there are precisely 2 coincidences among the 8 geometries of closed 3-manifolds, when these geometries are considered up to quasiisometry.

**Definition.** Let \( A \) be a finite set of semigroup generators for the group \( G \) and let \( \sigma : G \to A^* \) be a combing of \( G \). We say that \( \sigma \) has \( n \)th-power bounded growth if \( \ell(\sigma(g)) = O(|g|^n) \) as \( |g| \to \infty \), and we say that \( \sigma \) has sub-\( n \)th-power bounded growth if \( \ell(\sigma(g)) = o(|g|^n) \) as \( |g| \to \infty \).

The next result is proved by the methods of [EC] and [BGSS].

**Proposition 3.9.**

3.9.1 If the group \( G \) possesses an \( n \)th-power bounded combing, then \( G \) satisfies an \((n + 1)\)st-power isoperimetric inequality.\(^2\)

\(^2\)This means that the Dehn function \( f(k) \) [G2] for any finite presentation of \( G \) (which counts the maximum over all relations of length at most \( k \) of the minimal area of van Kampen diagrams for these relations) satisfies \( f(k) \leq p(k) \), where \( p \) is a polynomial of degree at most \( n + 1 \).
3.9.2 If the group $G$ satisfies a sub-$n^{\text{th}}$-power bounded combing, the $G$ satisfies a sub-$(n+1)^{\text{st}}$-power isoperimetric inequality. □

Example. The group $E$ of (2.1) of Raghunathan’s example satisfies the quadratic isoperimetric inequality. The group $\Gamma$ there is a cocompact lattice in $\text{SO}^+(2,m)$, so is combable with a linearly bounded combing. It follows from Theorem 3.11 that $E$ is combable with a linearly bounded combing. Proposition 3.9.1 then shows that $E$ satisfies the quadratic isoperimetric inequality.

Question. Does the integral 3-dimensional Heisenberg group $G$ possess a combing? By Proposition 3.9.2 and the fact that $G$ satisfies a sharp cubic isoperimetric inequality [ECH] [G2], we know that it does not possess a combing with subquadratic growth. Gromov sketches an argument that $G$ is combable in [Gr4]. Our analysis of that argument shows that it gives an asynchronous combing of the Heisenberg group $G$. In this connection, it is worth mentioning a result of M. Bridson [Br] which states that every group of the form $\mathbb{Z}^n \rtimes \phi \mathbb{Z}$ possesses an asynchronous combing.

4. Bicombinability of certain central extensions

In this section, we shall prove that a central extension

$$1 \to \mathbb{Z} \to E \to G \to 1,$$

which is defined by a finite valued 2-cocycle, is such that $E$ is bicombinable, provided that $G$ is combinable. We point out that although combinability is a quasiisometry invariant property, it is not known whether bicombinability is quasiisometry invariant; thus our result does not follow immediately from Theorem 3.1. We begin with a simple example, to show that bicombinability is a more subtle property than combinability.

Example. Let $G = F(x,y)$, the free group freely generated by $A = \{x,y\}$, and let $\phi \in \text{Aut}(G)$ be given by $\phi(x) = x^{-1}y$, $\phi(y) = x^{-1}$. One checks that $\phi$ is of order 3, so one can form the split extension $H = G \rtimes \phi \mathbb{Z}_3$, where $H$ is given by the presentation $\langle x, y, t \mid x^t = x^{-1}y, y^t = x^{-1}, t^3 = 1 \rangle$. The set of reduced words is a combing for $G$. One takes as generators for $H$ the set $B = \{x^{\pm 1}, y^{\pm 1}, t\}$. A combing of $H$ is given by the language $L \subset B^*$ given by

$$L = \{w(x,y)t^i \mid w \text{ is a reduced word in } G, \ 0 \leq i \leq 2\}.$$

However, the language $L$ is not a bicombing for $H$. To see this, note that one has the equation in $H$, $tx^{2n} = (x^{-1}y)^{2nt}$, for $n > 0$, and $x^{2n}$, $(x^{-1}y)^{2nt} \in L$. If $L$ were a bicombing, then there would exist $k > 0$ so that $d_{G,B}(x^{2n},(x^{-1}y)^i) \leq k$ for all $n \leq i \leq 2n$. But $H$ is quasiisometric to $G$, since it is a finite extension of $G$, and $d_{G,A}(x^{2n},(x^{-1}y)^i) = |x^{-2n}(x^{-1}y)^{i}G,A = 2(n+i)|$, which is unbounded, proving that $L$ is not a bicombing.

Note here that $H$ is word hyperbolic and hence bicombinable, but the “obvious” language $L$ fails to give the bicombing.
Theorem 4.1. If the central extension

\[ 1 \to \mathbb{Z} \to E \to G \to 1 \]

is defined by a finite valued 2-cocycle and if \( G \) has a bicombing, then \( E \) has a bicombing. In addition, if the bicombing of \( G \) is linearly bounded, then so is the bicombing of \( E \).

Proof. Let \( A \) be a finite set of semigroup generators for \( G \) and let \( \sigma : G \to A^* \) be a bicombing of \( G \) with bimodulus of continuity \( k \geq 1 \). Let \( \mathcal{L} = \sigma(G) \subset A^* \) be the associated language of the bicombing. We take as generator for the central kernel \( t \) and note that \( \mathcal{M} = \{ t^m \mid m \in \mathbb{Z} \} \) is a bicombinable language for \( \langle t \rangle \) (the modulus of bicontinuity here is 1). Let \( f : G \times G \to \mathbb{Z} \) be the finite valued 2-cocycle defining the extension; we assume that \( f \) is normalized, as in §3, and let \( |f(g, g')| \leq M \) with \( M \geq 1 \) (the symbol \( |f(g, g')| \) refers to the word metric on \( \langle t \rangle \)).

We recall that \( E \) is given as a set by \( \langle t \rangle \times G \), where the group law is \( (t^m, g) \cdot (t^{m'}, g') = (t^{m+m'}, f(g, g'), gg') \). We take as semigroup generators for \( E \) the set

\[ S = \{(t^\pm 1, 1), (t^u, a) \mid |u| \leq M, a \in A \}. \]

We have need of the next result; the construction of the generators \( A_i \) in the proof plays a crucial role in the sequel.

Lemma 4.2. For all \( g \in G \) we have

\[ |g|_{G, A} = |(1, g)|_{E, S}. \]

Proof. The projection \( E \to G \) is a contraction mapping for the word metrics, so \( |g|_{G, A} \leq |(1, g)|_{E, S} \).

For the reverse inequality, suppose that \( g = a_1a_2\ldots a_n \) with \( a_i \in A \). Let \( g_i = a_1a_2\ldots a_i \) for \( 1 \leq i \leq n \), \( g_0 = 1 \). Then one verifies that in \( E \) we have the identity \( (1, g_i+1) = (1, g_i)A_{i+1} \); here, \( A_{i+1} = (t^{-u_i+1}, a_i+1) \), where \( u_{i+1} = f(g_i, a_i+1) \). Thus \( |u_{i+1}| \leq M \) and \( A_i \in S \) for \( 1 \leq i \leq n - 1 \). Also note that \( (1, g) = (1, g_n) = A_1A_2\ldots A_n \). It follows that \( |g|_{G, A} = |(1, g)|_{E, S} \), and the lemma is established.

Returning to the proof of the theorem, if \( y \in \mathcal{L} \) is such that \( y = a_1a_2\ldots a_n \), with \( a_i \in A \), then we let \( \tilde{y} = A_1A_2\ldots A_n \in S^* \), with \( A_i \) constructed as in the proof of Lemma 4.2. Let \( \tilde{\mathcal{L}} = \{ \tilde{y} \mid y \in \mathcal{L} \} \). Observe that if we view \( \tilde{y} \) as a path \( \tilde{y}(t) \) in the Cayley graph for \( E, S \), then we have

\[ \tilde{y}(i) = (1, y(i)) \]

for integers \( i \); here \( y(t) \) is the path determined by \( y \) in the Cayley graph for \( G, A \).

Let \( \mathcal{M} = \{(t^m, 1) \mid m \in \mathbb{Z} \} \) and let \( \mathcal{N} = \mathcal{M} \cdot \tilde{\mathcal{L}} \subset S^* \). If \( x = t^m \in \mathcal{M} \), we denote by \( \tilde{x} = (t^m, 1) \in \mathcal{M} \). Thus every word in \( \mathcal{N} \) is of the form \( \tilde{x}\tilde{y} \), where \( x \in \mathcal{M} \) and \( y \in \tilde{\mathcal{L}} \). Using (4.1.1) we see that \( \mathcal{N} \) is a language which represents every element
of $E$ uniquely. We shall next show that $\tilde{\mathcal{N}}$ (or equivalently the associated section $E \to S^* \mathcal{N}$) is a bicombing.

Suppose we have $\tilde{x}y \cdot (t^u, a) \equiv \tilde{x}'y'$, where ‘$\equiv$’ means the words represent the same elements of $E$. Let $y = a_1a_2 \ldots a_n$, $y' = a'_1a'_2 \ldots a'_{n'}$ with $a_i, a'_i \in S$ and let $x = t^m$, $x' = t^{m'}$. We see that we have $m + u + f(\tilde{y}, a) = m'$, where $\tilde{y} \in G$ is the element of $G$ represented by $y$, and $\tilde{y} \cdot a = y'$ in $G$. It follows that $|m' - m| \leq 2M$ and, using the fact that $\mathcal{L}$ is a bicombing of $G$ with modulus of continuity $k$, we get $|y(t) - y'(t)|_{E, \mathcal{L}} \leq k$ for all $t \geq 0$.

**Lemma 4.3.** We have $d_{E, \mathcal{L}}(\tilde{y}(i), \tilde{y}'(i)) \leq 2M + k$ for all $i \geq 0$.

**Proof.** We have

$$d_{E, \mathcal{L}}(\tilde{y}(i), \tilde{y}'(i)) = d_{E, \mathcal{L}}((1, y(i)), (1, y'(i)))$$

$$= d_{E, \mathcal{L}}(1, (1, y(i))^{-1}(1, y'(i)))$$

$$= |(t^v, y(i)^{-1}y'(i))|_{E, \mathcal{L}} \quad \text{with} \quad |v| \leq 2M$$

$$\leq 2M + k,$$

where Lemma 4.2 was used in the last step, along with the triangle inequality.

In the words $\tilde{x}\tilde{y}$, $\tilde{x}'\tilde{y}'$ above, we do not begin to read the $\tilde{y}, \tilde{y}'$ segments at the same time, but these are at most $2M$ out of synchronization. Taking into account Lemma 4.3 to estimate the word difference of these words, we see that $4M + k$ bounds the word difference for $\tilde{x}\tilde{y}, \tilde{x}'\tilde{y}'$. A similar calculation for the right multiplication by a generator $(t^\pm, 1) \in S$ shows that $4M + k$ is a modulus of continuity for $\tilde{\mathcal{N}}$. Then one carries out the same calculation for left multiplication by the generators of $S$ to see that this is also a modulus of bicontinuity; this last calculation makes use of the data that $k$ is a modulus of bicontinuity for the language $\mathcal{L}$ for $G$.

From the construction of the language $\tilde{\mathcal{N}}$ for $E$, we see that the corresponding section $E \to S^* \mathcal{N}$ is linearly bounded if $\sigma : G \to A^*$ is linearly bounded. This completes the proof of Theorem 4.1

**Remark.** The result is of interest, since the existence of a linearly bounded bicombing on a group $E$ has implications for the structure of the group. The results of [GS3], Section 6, and [Sh] apply to this situation to show that translation numbers $(q, v)$ for elements of infinite order are strictly positive. This implies, among other things, that polycyclic subgroups of $E$ are virtually abelian.

**Remark.** It follows from Theorems 2.2 and 4.1 that the group $E$ in the central extension 2.1 is bicombing with a linearly bounded bicombing, thereby strengthening Corollary 3.7.

**Theorem 4.4.** If $M$ is a compact Seifert fibred manifold over a hyperbolic orbifold, then $\pi_1(M)$ is bicombable.
**Proof.** We write out the details in the case when $M$ is closed. The bounded case is treated similarly. If $X$ is the hyperbolic orbifold quotient of $M$, then by [Sc] there is a central extension of groups

\[(4.4.1) \quad 1 \to \mathbb{Z} \to \pi_1(M) \to \pi_1(X) \to 1,\]

where $\pi_1(X)$ denotes the orbifold fundamental group. Since $\pi_1(X)$ contains a closed orientable hyperbolic surface group $H$ as a subgroup of finite index (in effect, from Selberg’s lemma), we can pull back the central extension \((4.4.1)\) to $H$ to obtain a commutative diagram of central extensions

\[
\begin{array}{cccc}
1 & \to & \mathbb{Z} & \to & E & \to & H & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & \mathbb{Z} & \to & \pi_1(M) & \to & \pi_1(X) & \to & 1.
\end{array}
\]

It follows from Example 1.4, the fact that $H$ is word hyperbolic, hence bicable, and Theorem 4.1 that $E$ is bicombable.

Let $n$ be the index of $H$ in $\pi_1(X)$. It follows from transfer theory for the cohomology of groups that if $x$ denotes the class of the central extension \((4.4.1)\) in $H^2(\pi_1(X), \mathbb{Z})$, then $\text{Cor} \circ \text{Res} = n$, where $\text{Res}$ denotes the restriction map and $\text{Cor}$ denotes the corestriction map [Sc]. In this way we see that the cohomology class $nx$ is represented by a bounded cocycle. On the other hand, the class $nx$ is represented by the second row in the following commutative diagram of central extensions

\[
\begin{array}{cccc}
1 & \to & \mathbb{Z} & \to & \pi_1(M) & \to & \pi_1(X) & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & \mathbb{Z} & \to & E' & \to & \pi_1(X) & \to & 1.
\end{array}
\]

Since $\pi_1(X)$ is word hyperbolic and since the second row of \((4.4.3)\) is defined by a bounded cocycle, it follows from Theorem 4.1 that $E'$ is bicombable. On the other hand, $\pi_1(M)$ is of finite index in $E'$. Thus $\pi_1(M)$ is a quasiconvex subgroup of $E'$, and it follows from the results of [Sh] that $\pi_1(M)$ is bicombable. This completes the proof of the theorem.

### §5. More on combability

Recall from Section 3 that the notion of combability we are using involves a section $\sigma : G \to A^*$ possessing a modulus of continuity $k > 0$, where $A$ is a finite set of semigroup generators for the group $G$. Observe that we made no assumptions in general about the lengths of the words $\sigma(g)$, $g \in G$. This agrees with the definition of combability in [Ghys2] and [Sh], but is weaker than the notion adopted in [ECH]. As we showed in Proposition 3.8.1, a group which possesses a linearly bounded combing satisfies the quadratic isoperimetric inequality.
**Proposition 5.1.** If $G$ is a combable group, then $G$ has an exponential isoperimetric function.

*Proof.* Let $\sigma : G \to A^*$ be a combing of $G$, where $A$ is a finite set of semigroup generators and $k$ is a modulus of continuity for $\sigma$. Let $P$ be the finite presentation whose set of generators is $A$ and whose relators are all words of length at most $2k + 2$ in the generators which represent $1$ in $G$. We must show that there exists $C > 1$ such that if $w \in A^*$ is any word representing $1$ in $G$, then there exists a van Kampen diagram $D$ for $w$ with $\text{Area}(D) \leq C\ell(w)$.

With this in mind, we first find a van Kampen diagram $D'$ for $w$ by using the paths $\sigma_i := \sigma(w(i))$ in the Cayley graph and observing that $d(\sigma_{i-1}(j), \sigma_i(j)) \leq k$, for all $i, j$. Choose paths $A_{i,j}$ in the Cayley graph $\Gamma$ joining vertices $\sigma_{i-1}(j)$ and $\sigma_i(j)$ such that $\ell(A_{i,j}) \leq k$.

If $a_i$ denotes the $i$th letter of $w$, then the circuit $\sigma_{i-1}a_i\bar{\sigma}_i$ can be filled in with a diagram $D'_i$ of width at most $k$, and the diagrams $D'_{i-1}$ and $D'_i$ fit together, along the boundary paths labelled $\sigma_i$, to give the required diagram $D'$ for $w$; in this diagram, the vertices $\sigma_{i-1}(j)$ and $\sigma_i(j)$ are joined by paths $A_{i,j}$ constructed in the preceding paragraph.

Next, we hold $j$ fixed and consider the path $w_j := A_{1,j}A_{2,j} \ldots A_{n,j}$, where $n = \ell(w)$. Observe that the paths $A_{i,j}$ do indeed fit together since the end point $\sigma_i(j)$ of $A_{i,j}$ is the initial point of $A_{i+1,j}$. Observe also that $\ell(w_j) \leq kn$.

We can now visualize a null homotopy $H'$ of $w$ as a mapping of a rectangle $R'$, 3 sides of which are mapped to the base point $1$ in $\Gamma$ and the fourth side of which is mapped by $w$. Horizontal lines are mapped by the paths $\sigma_i$ and vertical lines are mapped by $w_j$, where $w = w_I$ for the largest value $I = \max_i \ell(\sigma_i)$. A schematic picture of the domain of $H'$ appears in Figure 1 below.
We now do surgery on the domain of $H'$. Observe that since $\ell(w_j) \leq kn$, we have the number of pairwise distinct possible words $w_j$ is at most $r^{kn}$, where $r = |A|$. Hence, if $I > r^{kn}$, then two such words, say $w_{j_0}$ and $w_{j_1}$, must be equal. We then cut out the subrectangle of $R'$ between these two vertical lines, to obtain a new homotopy with properties similar to $H'$. One continues doing surgeries until all the vertical lines are labelled by different words. The number of such words then will be at most $r^{kn}$ and each such word will be of length at most $kn$. Call the resulting homotopy $H$ and its domain the rectangle $R$. It follows that the area of $R$ is at most $knr^{kn}$. Finally we collapse the three sides of $R$ mapped to the base point to a single point, to obtain the desired diagram $D$ for $w$, and observe that Area$(D) \leq knr^{kn}$. This bound grows no faster than $C^n$ for suitable constant $C > 1$, completing the proof of the proposition.

There is one special case of interest where we can prove a combing is linearly bounded, and hence the group satisfies the quadratic isoperimetric inequality. We let $\sigma : G \to A^*$ be the combing as above with modulus of continuity $k > 0$. Let $L = \sigma(G) \subset A^*$, the associated language. We say that $L$ is prefix closed if every initial segment of a word in $L$ is also in $L$.

**Proposition 5.2.** Assume the language $L$ of the combing $\sigma : G \to A^*$ is prefix closed. Then $\sigma$ is linearly bounded, and consequently $G$ satisfies the quadratic isoperimetric inequality.
Proof. We let \( L = \sigma(G) \subseteq A^* \) and let \( k > 0 \) be the modulus of continuity. Let \( N \) be the number of vertices in the ball of radius \( k \) centered at 1 in the Cayley graph.

We claim that \( |\ell(\sigma(ga)) - \ell(\sigma(g))| \leq N \) for all \( g \in G, \ a \in A \). For assume the contrary, that for some \( g \in G, \ a \in A \) one has \( |\ell(\sigma(ga)) - \ell(\sigma(g))| \geq N + 1 \). We give the argument in the case where \( \ell(\sigma(ga)) - \ell(\sigma(g)) \geq N + 1 \). Let then \( n = \ell(\sigma(ga)) \) and let \( \sigma(ga) = a_1a_2\ldots a_n \) with \( a_i \in A \) and let \( m = \ell(\sigma(g)) \). Note that \( n \geq m + N + 1 \). We have \( |g^{-1}\sigma(ga)(i)|_{\mathcal{G},A} \leq k \) for all \( i \) satisfying \( m \leq i \leq n \). But there are at most \( N \) such word differences \( g^{-1}\sigma(ga)(i) \) for \( i \) in the indicated range. That implies there are two distinct values, say \( m \leq p \neq q \leq n \), such that \( g^{-1}\sigma(ga)(p) = g^{-1}\sigma(ga)(q) \), and consequently \( \sigma(ga)(p) = \sigma(ga)(q) \). But the assumption that \( L = \sigma(G) \) is prefix closed means that both of the words \( a_1a_2\ldots a_p \) and \( a_1a_2\ldots a_q \) are in \( L \), and these words represent the same element of \( G \). This contradiction (to \( \sigma \) being a section of the evaluation mapping) shows that \( |\ell(\sigma(ga)) - \ell(\sigma(g))| \leq N \).

An entirely analogous argument shows that \( |\ell(\sigma(g)) - \ell(\sigma(ga))| \leq N \), and we deduce that \( |\ell(\sigma(ga)) - \ell(\sigma(g))| \leq N \), verifying the claim.

Now it is an easy matter to prove from the last inequality by induction on \( |g| \) that \( \ell(\sigma(g)) \leq N|g| + B \), where \( B = \ell(\sigma(1)) \). It follows that \( \sigma \) is linearly bounded. The last statement of the proposition follows from Proposition 3.12.1, and the proof is complete.

Remark. If \( \sigma : G \to A^* \) were a combing of such a group \( G \) which did not satisfy the quadratic isoperimetric inequality (where \( A \) is a finite set of semigroup generators and \( k \) is a modulus of continuity), then we note that the language \( L := \sigma(G) \) cannot be regular.
Appendix:
Boundedness of the Characteristic Class
of Raghunathan’s Central Extension

by Domingo Toledo

This appendix, written at the request of S. M. Gersten, is an exposition of standard facts which were used in Section 2 of the preceding article.

Notation:
$G = \text{SO}^+(2, n)$ = the identity component of $\text{SO}(2, n)$; we assume $n \geq 3$ throughout this Appendix. This ensures that $G$ is simple (and that the group $K_0$ as defined below is indeed semi-simple).
$K = \text{SO}(2) \times \text{SO}(n)$ = the maximal compact subgroup of $G$. Note that $K$ is a deformation retract of $G$.
$K_0 = \text{SO}(n) = \text{the semi-simple part of } K$.
$X = G/K = \text{the symmetric space of } G$. The space $X$ is contractible and the group $G$ acts transitively on $X$.
$Y = G/K_0 = \text{the total space of the principal } \text{SO}(2)\text{-bundle } p : Y \to X$, where $p$ is the natural projection $G/K_0 \to G/K$. Thus $p : Y \to X$ is a $G$-homogeneous $\text{SO}(2)$-bundle over $X$.
$\hat{G} = \text{the covering group of } G \text{ corresponding to the subgroup } \pi_1(K_0) \cong \mathbb{Z}/2\mathbb{Z}$ of $\pi_1(G)$. Thus $\hat{G} \to G$ is an infinite cyclic cover.
$Z = \text{the kernel of the covering homomorphism } \pi : \hat{G} \to G$. Thus $Z$ is an infinite cyclic group.
$\hat{K}_0 = \text{the identity component of } \pi^{-1}(K_0)$. Note that $\hat{K}_0$ is a maximal compact subgroup of $\hat{G}$, and that $\pi$ maps $\hat{K}_0$ isomorphically onto $K_0$.
$\hat{Y} = \hat{G}/\hat{K}_0 = \text{the universal covering space of } Y$. Note that the natural projection $\hat{Y} \to Y$ is an infinite cyclic cover, and that $\hat{Y}$ is a contractible space on which $\hat{G}$ acts properly and transitively.
$\Gamma = \text{a discrete, torsion-free, cocompact subgroup of } G$.
$M = \Gamma\backslash X = \Gamma\backslash G/K = \text{a compact aspherical manifold locally isometric to } X$ and with fundamental group $\Gamma$, hence the Eilenberg-MacLane space $K(\Gamma, 1)$.
$N = \Gamma\backslash Y = \Gamma\backslash G/K_0 = \text{the total space of the principal } \text{SO}(2)\text{-bundle } p : N \to M$ induced from the projection $Y \to X$.
$E = \text{the subgroup } \pi^{-1}(\Gamma) \text{ of } \hat{G}$.

Thus there is a central extension

\[
(*) \quad 1 \to Z \to E \to \Gamma \to 1.
\]
This is the extension in the top row of (2.1) in the article.

**Lemma A1.** The central extension (\(\ast\)) is isomorphic to the central extension
\[
(\ast\ast) \quad 1 \rightarrow \mathbb{Z} \rightarrow \pi_1(N) \rightarrow \pi_1(M) \rightarrow 1,
\]
which arises from the exact homotopy sequence of the fibration \(p : N \rightarrow M\). Hence the characteristic class of (\(\ast\)) in \(H^2(\Gamma, \mathbb{Z})\) corresponds (possibly up to sign) to the Chern class of \(p\) in \(H^2(M, \mathbb{Z})\).

**Proof.** Observe that \(E\) acts freely on \(\hat{Y}\) and that \(E \backslash \hat{Y} = \Gamma \backslash Y = N\), hence \(N\) is the Eilenberg-MacLane space \(K(E, 1)\). From this it follows easily that the extensions (\(\ast\)) and (\(\ast\ast\)) are isomorphic. The statement about the correspondence of characteristic classes is standard and follows from the obstruction theoretic interpretation of the Chern class of the complex line bundle \(p\). (The proof of Lemma 3 in [Mi] contains all the essential details.)

Suppose now that \(P\) is a \(G\)-homogeneous principal \(U(1)\)-bundle over \(X\). Let \(x_0 \in X\) be the point corresponding to the coset of the identity element, and let \(P_0\) denote the fibre of \(P\) over \(x_0\). The action of \(G\) on \(P\) by bundle automorphisms restricts to a unitary action of \(K\) on \(P_0\). Let \(\rho_P : K \rightarrow U(1)\), called the isotropy representation of \(P\), denote the homomorphism which describes this action.

**Lemma A2.** The map that assigns to \(P\) the isotropy representation \(\rho_P\) is an isomorphism of the group (under tensor product) of isomorphism classes of \(G\)-homogeneous principal \(U(1)\)-bundles over \(X\) onto the group \(\text{Hom}(K, U(1))\) of continuous homomorphisms of \(K\) to \(U(1)\).

**Proof.** The assignment is clearly a group homomorphism. Its inverse can be constructed as follows. To the element \(\rho \in \text{Hom}(K, U(1))\) assign the bundle \(P_\rho = G \times U(1)\). By construction it is clear that \(\rho_{P_\rho} = \rho\). If \(P\) is a \(G\)-homogeneous line bundle and \(\phi : U(1) \rightarrow P_0\) is a given \(U(1)\)-equivariant isomorphism, then the map
\[
G \times U(1) \rightarrow P
\]
defined by
\[
(g, \theta) \mapsto g \cdot \phi(\theta)
\]
descends to a bundle isomorphism \(G \times U(1) \rightarrow P\). Hence \(P_{\rho_P} = P\), and the lemma is established.

Since \(\text{Hom}(K, U(1)) = \text{Hom}(\text{SO}(2), U(1)) \cong \mathbb{Z}\), it follows that the group of \(G\)-homogeneous line bundles over \(X\) is infinite cyclic; and it is generated by the class of the line bundle \(p : Y \rightarrow X\), since its isotropy representation, being an isomorphism, is clearly a generator of \(\text{Hom}(\text{SO}(2), U(1))\).
**Proposition A3.** Let \( c \in H^2(M) \cong H^2(\Gamma) \) be the characteristic class of the complex line bundle \( p : N \to M \) (equivalently, \( c \) is the characteristic class of the extension \((*)\)). Then \( c \) is represented by a bounded real cochain.

**Proof.** Observe that \( M \) is a Kähler manifold, where the complex structure and metric are those that descend from the corresponding \( G \)-invariant complex structure and metric on \( X \) (cf. Chapter VIII of [He]). Moreover \( X \) can be realized as a bounded domain in \( \mathbb{C}^n \) and the metric then corresponds to the Bergmann metric (cf. [Ca], [He]).

Using this Kähler structure on \( M \), let \( \Omega \) denote the circle bundle of unit vectors associated to the canonical bundle (= determinant of the holomorphic cotangent bundle of \( M \)). This lifts to a \( G \)-homogeneous \( U(1) \) bundle on \( X \). Hence this lift is a multiple of \( p : Y \to X \). It follows that \( \Omega \) is a multiple of the line bundle \( p : N \to M \). But by definition of the Bergman metric, \( c_1(\Omega) \) is represented (up to a positive normalizing factor) by the Kähler \( \omega \) form of the Bergman metric, (cf., e.g., the proof of Cor. 4, Chap. V, n 6, of [We]). It follows that the Chern class of \( p : N \to M \) is a nonzero real multiple of \( \omega \). But it is known that the cohomology class \( \omega \) is bounded. In fact an explicit bound \( 2\pi \) follows from the main theorem stated in [DT], and its boundedness also follows from the various references discussed in its introduction, e.g., Gromov’s general theorem on boundedness of characteristic classes of flat bundles [Gr1], §1.3. Therefore \( c \) is a bounded real cohomology class.

**Remarks:**

1. Strictly speaking, the proof given in [DT] treats all the classical Hermitian symmetric spaces except for those of the group \( \text{SO}(2, n) \). Thus it is appropriate to explain why it also applies to \( \text{SO}(2, n) \):

   First, using the special local isomorphism between \( \text{SO}(2, 4) \) and \( \text{SU}(2, 2) \), cf. [Ca], [He], we see that the elementary proof given there of the value \( 2\pi \) for the supremum norm of the class of \( \omega \) applies to the group \( \text{SO}(2, 4) \). But this immediately implies that the Kähler class is bounded and of supremum norm \( 2\pi \) for the groups \( \text{SO}(2, n) \), \( n \geq 3 \) (and also for \( n = 2 \) which is not relevant to this paper), for the following reason. If \( n \geq 5 \), the same bound has to hold because any geodesic triangle in the symmetric space for \( \text{SO}(2, n) \) is contained in a totally geodesic embedding of the symmetric space of \( \text{SO}(2, 4) \). This is true for the simple reason that, since the symmetric space of \( \text{SO}(2, n) \) is the set of 2-planes in \( \mathbb{R}^{n+2} \) on which a form of signature \( (2, n) \) is positive definite, any three such planes are contained in a single \( \mathbb{R}^6 \) on which the form has signature \( (2, 4) \), hence any three points are contained in a symmetric space for \( \text{SO}(2, 4) \). For \( \text{SO}(2, 3) \) the bound is clear since its symmetric space is a totally geodesic subspace of that of \( \text{SO}(2, 4) \). That these upper bounds are sharp can be shown by evaluating on suitable discrete subgroups, in a manner similar to that Section 3 of [DT].

2. In the above discussion we could take \( G \) to be the group of automorphisms of any irreducible Hermitian symmetric space. Namely, its maximal compact subgroup \( K \) is of the form \( \text{SO}(2) \times K_0 \) for a semi-simple group \( K_0 \), hence the extension \((*)\) can
be defined using the infinite cyclic covering corresponding to the subgroup $\pi_1(K_0)$, and its characteristic class is represented by a bounded real cocycle. Raghunathan informs us that it is now known that $E$ is not residually finite whenever $G$ has real rank at least two, with the possible exception of some lattices in one of the exceptional Hermitian symmetric spaces. Observe that a proof for $\text{Sp}(2n, \mathbb{R})$ is sketched in a note added at the end of [Ra1]. Thus the phenomenon of Section 2 of the preceding paper is not restricted to the group $\text{SO}(2, n)$. 
BOUNDED COCYLES

References


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