PRESERVATION AND DISTORTION OF AREA IN FINITELY PRESENTED GROUPS

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Abstract. If $K = G \rtimes_\phi \mathbb{Z}$ where ϕ is a tame automorphism of the 1-relator group G, then the combinatorial area of loops in a Cayley graph of G is undistorted in a Cayley graph of K. Examples of distortion of area in fibres of fibrations over the circle are given and a notion of exponent of area distortion is introduced and studied. The inclusion of a finitely generated abelian subgroup in the fundamental group of a compact 3-manifold does not distort area.

1. Introduction.

A theorem of Sullivan's states that if M is a 3-manifold and $\mathcal{F}$ is a codimension 1 foliation on M which is transversely oriented and possessing the property that through every leaf there passes a transverse closed curve, then each leaf of $\mathcal{F}$ is quasi-area minimizing [Ha]. It follows that in the fibration $M \to S^1$, where M is a 3-dimensional Sol manifold [Sc], the fibre is a minimal surface in M for a suitable Riemannian metric on M. The algebraic counterpart to this fact is that in the split extension $G = \mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$, where ϕ is a hyperbolic matrix in $\mathrm{Gl}_2(\mathbb{Z})$ (so $\mathrm{Trace}(\phi)^2 > 2$), the combinatorial area of loops in a finite presentation for the $\mathbb{Z}^2$-fibre is preserved, up to a constant multiple, in a finite presentation for G. This fact is remarkable because the word metric of the $\mathbb{Z}^2$-fibre is exponentially distorted in the split extension (*cf.* [Gr2]), but the area is undistorted in a sense made precise in §2 below. In this article we shall prove an algebraic result which generalizes this fact about Sol manifolds.

Definition. If G is the group of the finite presentation $\mathcal{P}$, we say that an automorphism ϕ of G is $\mathcal{P}$-tame if ϕ lifts to an automorphism of the free group on the generators of $\mathcal{P}$ preserving the normal subgroup of relations.

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Theorem A. *If $\mathcal{P}$ is a finite 1-relator presentation of the group G and if ϕ is a $\mathcal{P}$-tame automorphism of G, then there exists $C > 0$ so that for all loops w in the Cayley graph of $\mathcal{P}$ one has*

$$Area_{\mathcal{P}}(w) \leq C \ Area_{\mathcal{Q}}(w),$$

where $\mathcal{Q}$ is a finite presentation for $G \rtimes_\phi \mathbb{Z}$.

The area $\text{Area}_{\mathcal{P}}(w)$ of a loop w in a finite presentation $\mathcal{P}$ is the minimal number of 2-cells in a van Kampen diagram for w [LS], or, expressed more algebraically, the minimal number of products of conjugates of defining relators or their inverses needed to freely represent the label of the loop w in the free group on the generators of $\mathcal{P}$. The fact about $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ quoted above follows from Theorem A for any $\phi \in \text{Gl}_2(\mathbb{Z})$ because, taking the 1-relator presentation $\mathcal{P} = \langle x, y \mid [x, y] \rangle$ for $\mathbb{Z}^2$, the automorphism ϕ lifts to the free group $F(x, y)$, and all automorphisms of $F(x, y)$ map the commutator $[x, y]$ into a conjugate of itself or its inverse (a fact due to Nielsen, which is evident from the fact that every automorphism of $F(x, y)$ is induced by a homeomorphism of the punctured torus, which necessarily preserves the boundary circle).

The analog of Theorem A holds with weak area replacing area, where the weak area of a loop in the Cayley graph is the minimal area of an orientable surface diagram with one boundary component, where that unique boundary component is mapped to the given loop.

At the Dimacs conference on geometric group theory, I proved that if H is a finitely presented subgroup of the finitely presented group G, where G is either of cohomological dimension 2 or where G admits a finite *diagrammatically aspherical* presentation $\mathcal{P}$ (so all spherical diagrams in $\mathcal{P}$ are reducible [LS]), then weak area is preserved (up to constant multiple) in considering a loop in the Cayley graph finite presentation of H as one in the Cayley graph of a finite presentation for G. This had the consequence that finitely presented subgroups H of hyperbolic groups G were hyperbolic, provided either G was of cohomological dimension 2 or G was a hyperbolic small cancellation group. In my article [Ge1] I also gave higher dimensional examples where area was not preserved in passing from H to G. One of my aims in writing this article is to give some higher dimensional examples of a group H where area (resp. weak area) is undistorted in imbedding H as a subgroup of a group G for nontrivial reasons (an example of a trivial reason would be that H was a retract of G).

In Theorem 5.1 I prove that if $K = G \rtimes_\phi \mathbb{Z}$ where G is finitely presented, then the area distortion of the subgroup G of K is at most an exponential of an isoperimetric function for K. In §4 I recall M. Bridson's example of a finitely presented subgroup H of an automatic group G where the area for the pair (G, H) is exponentially distorted and I show that the word metrics are undistorted for this pair of groups. Thus, and it is important to emphasize this fact, I produce examples $K = G \rtimes_\phi \mathbb{Z}$ with G finitely presented where the inclusion $G < K$ does not distort length but where area is distorted (Bridson's example, §4 below) and examples where length is distorted but where area is undistorted (the Sol groups mentioned above, where the first proof of lack of distortion of area followed from Sullivan's theorem). Otherwise stated, area distortion and length distortion in fibrations over the circle are *independent phenomena*.[1] In §7 I define the notion of geometric property of a pair consisting of finitely generated subgroup of a finitely generated group and I give several examples. Of particular interest in this connection is the *exponent of area distortion* $\alpha(G, H)$ for the inclusion of a finitely presented subgroup H in a finitely presented group G, when H has a polynomial isoperimetric function of degree d. A key inequality $1 \leq \alpha(G, H) \leq d$ follows from Proposition 5.4, which states that the area distortion is bounded by a polynomial of degree d under these hypotheses. In §8 I establish the *suspension theorem*, which states that if $\phi \in \mathrm{Gl}_n(\mathbb{Z})$ is of infinite order, then the exponent of area distortion of the torus bundle over the circle with monodromy $\phi \oplus 1 \in \mathrm{Gl}_{n+1}(\mathbb{Z})$ is strictly greater than 1. In §9 I give an example of an automatic subgroup H of an automatic group G where the exponent of area distortion is a number in the open interval $(1, 2)$. In §10 I consider central extensions and prove that the inclusion of a $\mathbb{Z}^2$ subgroup in the fundamental group of a closed geometric 3-manifold does not distort area. In §11 I extend the results of §10 to show, making use of results of [KL1],[KL2], that the inclusion of a finitely generated abelian subgroup in the fundamental group of a compact 3-manifold does not distort area. This last result can be viewed as a generalization of Sullivan's theorem, the genesis of this work, since the latter result implies that there is no area distortion in the inclusion $\mathbb{Z}^2 < \mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ when ϕ is a hyperbolic matrix, *i.e.* for the inclusion of the torus fibre in a Sol manifold. I grouped together in §12 several open questions suggested by this work.

I am grateful to Lee Mosher for telling me about Sullivan's theorem and for expressing interest in examples where area is distorted and undistorted in extensions.[2] Mladen Bestvina and Mark Feighn provided me with the crucial Example 6.5 from the theory of free group automorphisms which enabled me to give the first example of a bundle over the circle with fibre the n-dimensional torus and having PV monodromy where area is distorted in including the fibre in the total space. I profited from discussions with Domingo Toledo and Andrejs Treibergs. Finally John Stallings suggested to me the proof of Lemma 10.3.

2. Area distortion.

Definition. Let $\mathcal{P}$ be a subpresentation of the finite presentation $\mathcal{Q}$ (so each generator (resp. relator) of $\mathcal{P}$ is a generator (resp. relator) of $\mathcal{Q}$) with the property that the group G of the presentation $\mathcal{P}$ injects in the group K of $\mathcal{Q}$. For each circuit w

[1]A notion of length distortion is adopted in [Gr2] which is, up to a linear factor, the one adopted in [Fa] and here. Although the arguments of [Gr2], *e.g.*, that for Corollary 3.I'$_2$ p. 65, are presented in terms of the notion of length distortion, the conclusions only follow if one interprets them in terms of our notion of area distortion. One of our goals in this paper is to make the distinction explicit.

[2]At the same party in Nashville where Mosher explained Sullivan's theorem to me, Martin Bridson explained his example of area distortion in automatic groups discussed below in §4.

in the Cayley graph of $\mathcal{P}$ there is defined $\mathrm{Area}_{\mathcal{P}}(w)$ and $\mathrm{Area}_{\mathcal{Q}}(w)$, and the relation $\mathrm{Area}_{\mathcal{P}}(w) \geq \mathrm{Area}_{\mathcal{Q}}(w)$ holds (we only assumed $\mathcal{P}$ to be a subpresentation of $\mathcal{Q}$ to emphasize that area decreases in this general setting). We define the area distortion function $f : \mathbb{N} \to \mathbb{N}$ by the rule

$$f(n) = \max_{\mathrm{Area}_{\mathcal{Q}}(w) \leq n} \mathrm{Area}_{\mathcal{P}}(w),$$

where w runs over circuits in the Cayley graph of $\mathcal{P}$. It is not immediately clear that this number is finite, since arbitrarily long words can have a given small area. However one first considers edge loops w such that a minimal van Kampen diagram D for w in $\mathcal{Q}$ has given area n and such that D is a topological disc. This gives a bound depending on n on $\ell(w)$ and hence guarantees only finitely many orbits of such w under the action of G. Then one reduces the case of a general loop w to this special case (*cf.* the proof of Proposition A below).

If the function f is linearly bounded, *i.e.* if there exists a constant C so that for all w the inequality $\mathrm{Area}_{\mathcal{P}}(w) \leq C \cdot \mathrm{Area}_{\mathcal{Q}}(w)$ holds, then we say that the inclusion $G < K$ does not distort area (or, more loosely, we say that the inclusion $G < K$ "preserves" area).

Similarly, we can define the distortion function for weak area, by using orientable surface diagrams instead of van Kampen diagrams, and it makes sense to say the inclusion $G < K$ does not distort weak area (or $G < K$ preserves weak area). As in [Ge1] one must observe the caveat that surface diagrams for the presentation $\mathcal{P}$ take their values in the universal cover of the 2-complex canonically associated to $\mathcal{P}$.

These definitions are in analogy with the definition of distortion for the word metrics [Fa][Gr2]. As in the case of the word metrics, changing the presentations $\mathcal{Q}, \mathcal{P}$ only changes the area distortion (resp. weak area distortion) function at most by an affine change of variables in domain and range, together with the addition of a linear term (*cf.* [Fa]). In particular the statement that $G < K$ does not distort area is independent of presentations.

Examples.

2.1. If H is a finitely presented subgroup of the word hyperbolic group G, then it is easy to see that H is word hyperbolic if the inclusion $H < G$ does not distort area. This contrasts with the theorem that the finitely generated subgroup H of the word hyperbolic group G is quasi-convex iff the inclusion $H < G$ does not distort the word metric.

2.2. The main technical results of [Ge1] are that if G is a finitely presented subgroup of the finitely presented group K and if either K of cohomological dimension 2 or K admits a finite diagrammatically aspherical presentation, then the inclusion $G < K$ does not distort weak area.

In view of the interest in the question whether finitely presented subgroups of word hyperbolic groups are hyperbolic, we include the following elementary result, which provides a converse for the first assertion of Example 2.1.[3]

Proposition A. *If the word hyperbolic group G is a subgroup of the finitely presented group K, then the inclusion $G < K$ does not distort area.*

Proof. Choose a finite presentation $\mathcal{Q}$ for K which includes a subpresentation $\mathcal{P}$ for the subgroup $G < K$, and let M be the length of the longest relator of $\mathcal{Q}$. Consider first the case when some minimal van Kampen diagram D in $\mathcal{Q}$ for the loop w in Γ is a topological disc. In this case each edge of ∂D is incident with a face and at most M edges are incident with the same face. It follows that $\mathrm{Area}_{\mathcal{Q}}(w) \geq \frac{\ell(w)}{M}$. Since H is hyperbolic there exists $C > 0$ so that $\mathrm{Area}_{\mathcal{P}}(w) \leq C\ell(w)$, so $\mathrm{Area}_{\mathcal{Q}}(w) \geq \frac{1}{CM}\mathrm{Area}_{\mathcal{P}}(w)$.

Consider now a general loop w in Γ and let D be a minimal area van Kampen diagram for w in $\mathcal{Q}$. Let $D_1, D_2, \ldots, D_m$ be the maximal disc components of D and let w_i be the boundary label of D_i. Note that w_i are loops in Γ and that D_i is a minimal van Kampen diagram for w_i in $\mathcal{Q}$. Then we have

$$\begin{aligned}\mathrm{Area}_{\mathcal{Q}}(w) &= \sum_i \mathrm{Area}(D_i) = \sum_i \mathrm{Area}_{\mathcal{Q}}(w_i)\\ &\geq \sum_i \frac{1}{CM}\mathrm{Area}_{\mathcal{P}}(w_i) \geq \frac{1}{CM}\mathrm{Area}_{\mathcal{P}}(w).\end{aligned}$$

It follows that the inclusion $G < K$ does not distort area, and the proof is complete.

I shall digress a bit to explain the significance of area distortion, from the point of view of geometric group theory. Let $\mathcal{P}$ be a finite presentation for the group G and let H be a finitely generated subgroup of G. Assume for convenience that the set of generators of $\mathcal{P}$ contains a subset which generates H. In this setting, the Cayley graph Γ for $\mathcal{P}$ contains a subgraph Δ which is a Cayley graph for H.

Let us say that a loop w in Δ is null homotopic in a δ-neighborhood (for the word metric of Γ) $N = N_\delta(\Delta)$ of Δ, $0 < \delta < \infty$, if there exists a van Kampen diagram for w in $\mathcal{P}$ whose image (when lifted to the universal cover of the 2-complex canonically associated to $\mathcal{P}$) is contained in N.

Proposition B. *In this situation the following assertions hold.*

(1) *The subgroup H is finitely presented iff there exists a δ-neighborhood N of Δ such that every loop in Δ is null homotopic in N.*
(2) *The subgroup H is finitely presented and in addition the inclusion $H < G$ does not distort area iff there exist $C > 0$ and a δ-neighborhood N of Δ so that for every loop w in Δ some van Kampen diagram D for w in $\mathcal{P}$ has image contained in N and furthermore*

$$\mathrm{Area}(D) \leq C \cdot \mathrm{Area}_{\mathcal{P}}(w).$$

[3]For many years it was not realized that this was a question of interest, because an example is sketched in [Gr1] p. 126 of a finitely presented subgroup of a word hyperbolic group which is not hyperbolic. This example was recently shown to break down by M. Bestvina [Be].

The proof is an exercise in the by now standard techniques of geometric group theory (*cf.* [ECHLPT]).

Suppose now that $\mathcal{P} = \langle x_1, x_2, \dots, x_r \mid R_1, R_2, \dots, R_s \rangle$ is a finite presentation for the group G and $\phi_\star$ is an automorphism of the free group on the generators preserving the normal subgroup of relations. We define the complexity

$$c_{\mathcal{P}}(\phi_\star) = \sum_{i=1}^{s} \mathrm{Area}_{\mathcal{P}}(\phi_\star(R_i)) - s.$$

Note that $c_{\mathcal{P}}(\phi_\star) = 0$ iff $\phi_\star$ maps permutes the free conjugacy classes of the relators R_i up to inversion.

We recall the following result [Ra] Theorem 2.

Theorem (Rapaport). *Let ϕ be an automorphism of the finitely presented group G. Then there is a finite presentation $\mathcal{P}$ of G so that ϕ is $\mathcal{P}$-tame.*

Definition. Suppose that ϕ is an automorphism of the finitely presented group G. We define the complexity $c(\phi)$ to be the minimum of $c_{\mathcal{P}}(\phi_\star)$, where $\mathcal{P}$ is a finite presentation for G and $\phi_\star$ lifts ϕ, in the sense that it is an automorphism of the free group on the generators of $\mathcal{P}$ which preserves the normal subgroup of relations and induces ϕ on the quotient group G. By Rapaport's result, the complexity of an automorphism ϕ is defined and is a nonnegative integer.

Examples.

2.3. If $\psi : F \to F$ is an automorphism of the finitely generated free group F which stabilizes the set of cyclic words $R_1, R_2, \dots, R_s$, then ψ induces an automorphism ϕ of the quotient $G = F/N$, where N is the normal closure of $\{R_1, R_2, \dots, R_s\}$, and $c(\phi) = 0$.

2.4. If G is a finitely presented group and ϕ is an automorphism of G, then the complexity $c(\phi)$ depends only on the class of ϕ in the outer automorphism group of G. This is clear from the fact that inner automorphisms can always be lifted.

2.5. Let $\mathcal{P}$ be a 1-relator presentation for the group G and let ϕ be a $\mathcal{P}$-tame automorphism of G. Then $c(\phi) = 0$. For suppose ψ is a free group automorphism lifting ϕ and preserving the normal closure of the relator R. A theorem of Magnus states that ψ takes R into a conjugate of itself or its inverse [LS] page 106 Proposition 5.8. But this means the complexity of ψ, and hence that of ϕ, is 0.

2.6. A familiar example where Example 2.5 applies is the fundamental group G of a closed orientable surface S. By the theorem of Dehn and Nielsen (*cf.* [De] Appendix), every element α of $\mathrm{Out}(G)$ is induced by a homeomorphism of the singly punctured surface $S - \{x\}$. This implies that α is represented by a $\mathcal{P}$-tame automorphism of G, where $\mathcal{P}$ is the usual 1-relator presentation for G (*cf.* [MKS]) (the relator comes from compactifying $S - \{x\}$ to a manifold with boundary $\widehat{S}$ and observing that every homeomorphism of $S - \{x\}$ is isotopic to one which extends to a homeomorphism of $\widehat{S}$). Since inner automorphisms always lift, it follows that every element ϕ of $\mathrm{Aut}(G)$ is $\mathcal{P}$-tame and hence $c(\phi) = 0$ by Example 2.5 above.

2.7. Let $G = A \star_C B$, where $C = \langle t\rangle$ is infinite cyclic. A Dehn twist is an automorphism ϕ of G of the form $\phi(a) = a$, $a \in A$, and $\phi(b) = tbt^{-1}$, $b \in B$. There is an analogous notion of Dehn twist for the HNN case $G = A\star_C$, where C is infinite cyclic [RS2]. It is easy to check that $c(\phi) = 0$ for a Dehn twist ϕ. Similar remarks apply to composites of Dehn twists with respect to a fixed splitting of a group over $\mathbb{Z}$.

We can now state our main result.

Theorem B. *Suppose that ϕ is an automorphism of the finitely presented group G and $K = G \rtimes_\phi \mathbb{Z}$. If $c(\phi) = 0$, then the inclusion $G < K$ does not distort area (resp. weak area).*

The proof depends on a lemma.

Lemma. *Let $\mathcal{P} = \langle x_1, x_2, \dots, x_r \mid R_1, R_2, \dots, R_s\rangle$ be a finite presentation for the group G and let ψ be an automorphism of the free group $F = F(x_1, x_2, \dots, x_r)$ preserving the normal closure N of the relators and such that $c_{\mathcal{P}}(\psi) = 0$. Then for all $w \in N$ we have $\mathrm{Area}_{\mathcal{P}}(\psi(w)) = \mathrm{Area}_{\mathcal{P}}(w)$ and similarly the weak area of $\psi(w)$ is equal to the weak area of w.*

Proof. We have already observed that ψ permutes the conjugacy classes of the cyclic words R_i up to inversion. If $w = \prod_{i=1}^n R_{j_i}^{\epsilon_i u_i}$, where $\epsilon_i = \pm 1$ and $u_i \in F$, then $\psi(w) = \prod_{i=1}^n \psi(R_{j_i})^{\epsilon_i \psi(u_i)}$, and we see that $\psi(w)$ is the product of n conjugates of the defining relators or their inverses. It follows that $\mathrm{Area}_{\mathcal{P}}(\psi(w) \le \mathrm{Area}_{\mathcal{P}}(w)$. Applying the same argument to ψ^{-1} establishes the opposite inequality, so the areas are equal.

We proceed now to argue for weak area. However the weak area of w is minimum number of factors needed to express w as a product of conjugates of the defining relators or their inverses modulo the subgroup $[N, N]$. Since $\psi(N) = N$, we see again that ψ preserves weak area as well.

Proof of Theorem B. Let $\mathcal{P} = \langle x_1, x_2, \dots, x_r \mid R_1, R_2, \dots, R_s\rangle$ be a finite presentation for G such that the automorphism ϕ of G lifts to a free group automorphism ψ of the generators preserving the normal subgroup of relations and such that $c_{\mathcal{P}}(\psi) = 0$. A presentation $\mathcal{Q}$ for $K = G \rtimes_\phi \mathbb{Z}$ is given by

$$\mathcal{Q} = \langle x_1, x_2, \dots, x_r, t \mid R_i,\ tx_jt^{-1} = \psi(x_j);\ 1 \le i \le s,\ 1 \le j \le r\rangle.$$

Let X, Y be the canonical 2-complexes for $\mathcal{P}, \mathcal{Q}$ respectively, and note that X is a subcomplex of Y. There is a map $\rho : Y \to S^1$ corresponding to the projection $K \to \mathbb{Z}$ in the short exact sequence

$$1 \to G \to K \to \mathbb{Z} \to 1$$

which collapses X to the base point of S^1 and wraps the 1-cell corresponding to the generator t homeomorphically once around the circle. The 2-cells corresponding to the relators $tx_j t^{-1}\psi(x_j)^{-1}$ get flattened by ρ onto their t-sides, thinking of the 2-cell as a rectangle with the t-sides vertical. Choose a point $p \in S^1$ which is a regular value for the map ρ (any point other than the base point will do).

Suppose now that w is an edge loop in $\widetilde{X} \subset \widetilde{Y}$, the universal covers of X and Y, where the imbedding $\widetilde{X} \subset \widetilde{Y}$ is determined by a choices of base points in $\widetilde{X}, \widetilde{Y}$. We want to show that $\mathrm{Area}_{\widetilde{X}}(w) = \mathrm{Area}_{\widetilde{Y}}(w)$, where we note that the inequality $\mathrm{Area}_{\widetilde{X}}(w) \geq \mathrm{Area}_{\widetilde{Y}}(w)$ always holds.

Let $f : D \to \widetilde{Y}$ be a reduced van Kampen diagram for w in $\widetilde{Y}$, let $\pi : \widetilde{Y} \to S^1$ be the composition of the covering map $\widetilde{Y} \to Y$ with $\rho : Y \to S^1$, and let $\Gamma = (\pi \circ f)^{-1}(p)$. By transversality, Γ is a 1-manifold properly imbedded in D and since w maps to the base point of S^1 under π, it follows that Γ does not meet the boundary of D. Thus Γ is the disjoint union of a finite number of simple closed curves in the interior of D. Let Δ be an innermost simple closed curve of Γ, which exists since D is planar, and let U be the union of all cells of D which meet Δ. Observe that U is an open annulus. Let the labels in F of the inner and outer boundary components of U be u_0 and u_1 respectively and observe that we have either $\psi(u_0) = u_1$ or $\psi(u_1) = u_0$ in F, depending on the orientation of the transverse t-arc to Δ in U. Assume for definiteness that $\psi(u_0) = u_1$; the other case is argued similarly. By the Lemma, $\mathrm{Area}_{\mathcal{P}}(u_0) = \mathrm{Area}_{\mathcal{P}}(u_1)$. Since u_0 is the label of the inner boundary component of U, it bounds a subdiagram D_0 of D with values entirely in $\widetilde{X}$. Thus we may excise D_0 and U from D and replace them by a minimal van Kampen diagram for u_1 in $\widetilde{X}$. The area is decreased in the process by the number of 2-cells of U.

Proceeding in this manner, we can in a finite number of such surgeries replace D by a van Kampen diagram for D' for w in $\widetilde{X}$ having smaller area than D. It follows that the minimal area diagram for w must have Γ empty, and hence $\mathrm{Area}_{\widetilde{X}}(w) \leq \mathrm{Area}_{\widetilde{Y}}(w)$. Combined with the earlier inequality, this shows that $\mathrm{Area}_{\mathcal{P}}(w) = \mathrm{Area}_{\mathcal{Q}}(w)$.

Proceed now to the argument for weak area; again the weak area is only reduced in passing from $\widetilde{X}$ to $\widetilde{Y}$, so we must show the opposite inequality. Let $f : S \to \widetilde{Y}$ be an orientable surface diagram for the edge loop w in $\widetilde{X}$ and let $\Gamma = (\pi \circ f)^{-1}(p)$. The difficulty here is that a simple closed curve component of Γ need not separate S, so the innermost circle argument above fails and must be replaced by a different argument. Since the map $\pi : \widetilde{Y} \to S^1$ factors through the regular $\mathbb{Z}$-cover $\widehat{Y}$ of Y corresponding to the homomorphism $K \to Z$ above, we can refine the t-labels of edges in the diagram S to labels t_n, $n \in \mathbb{Z}$, corresponding to the composition of $f : S \to \widetilde{Y}$ with the covering map $\widetilde{Y} \to \widehat{Y}$. Here t_0 is the lift of the edge t of Y at the base point of $\widehat{Y}$ and t_n is the result of applying the n^{th} power of the preferred generator of the deck transformation group to the edge t_0.

If Γ is nonempty, then by compactness of S, there exists a maximal index $|N|$ such that t_N appears as a label in the diagram S. The corresponding boundary components of the annular neighborhoods of circles in Γ map into copies of $\widetilde{X}$ at the N^{th} level (that is, these copies of $\widetilde{X}$ map to the integer $N \in \rtimes$ under the lifted map $\widetilde{Y} \to \widehat{Y} \to \rtimes$). We shall partition these level N boundary components into equivalence classes as follows. Two such components are said to be equivalent if they can be connected in S without leaving level N; that is, they must be connected in a portion of the 1-skeleton $S^{(1)}$ of the domain of f without crossing a t-edge. Now connect up the components in each equivalence class by arcs at level N to get one circuit $\mathbb{Z}$ for each equivalence class. The key point to observe is that $\mathbb{Z}$ separates S; in effect, this comes to the fact that N separates $\rtimes$. Now we use the Lemma to observe that the label u of $\mathbb{Z}$ has the same weak area in $\mathcal{P}$ as $\psi^{\pm 1}(u)$. So surgery on S can be done to reduce the number of faces of S by the number of faces in the annuli corresponding to one equivalence class. This completes the argument for Theorem B in the case of weak area.

Theorem A of the Introduction is an immediate consequence of Theorem B and Example 2.5 above.

If M^3 is a closed 3-manifold which fibres (smoothly, say) over the circle with fibre F, then the inclusion $\pi_1(F) \to \pi_1(M)$ preserves area and weak area. This follows from Example 2.6 and Theorem A or, in the case of area, from Sullivan's theorem. In dimension $n \geq 4$, it is unknown whether a closed aspherical n-manifold M can fibre smoothly over S^1 if $\pi_1(M)$ is word hyperbolic. However we have the following partial result.

Corollary 1. *Suppose that for some $n \geq 4$ the closed aspherical n-manifold M fibres smoothly over S^1 with fibre the (closed aspherical) manifold N and suppose that $\pi_1(M)$ is word hyperbolic. Let $\phi \in \operatorname{Out}(\pi_1(N))$ be the monodromy in the homotopy exact sequence*

$$1 \to \pi_1(N) \to \pi_1(M) \to \mathbb{Z} \to 1.$$

Then $c(\phi) > 0$ and the inclusion $\pi_1(N) \to \pi_1(M)$ does not preserve weak areas.

Proof. It follows from [Be] Theorem 1 that $\pi_1(N)$ is *not* word hyperbolic under these hypotheses. However, if $c(\phi) = 0$, then the inclusion $\pi_1(N) \to \pi_1(M)$ would preserve weak area by Theorem B. But this would imply $\pi_1(N)$ was word hyperbolic [Ge1].

Corollary 2. *Suppose that $K = G \rtimes_\phi \mathbb{Z}$, where G has a finite 1-relator presentation $\mathcal{P}$ and $\phi \in \operatorname{Aut}(G)$ is $\mathcal{P}$-tame. If K is word hyperbolic, then G is also word hyperbolic.*

Proof. It follows from Example 2.5 that $c(\phi) = 0$, so weak area is undistorted by the inclusion $G < K$. Since K satisfies the linear isoperimetric inequality, it follows that G does also. It follows from [Ge1] that G is word hyperbolic.

Question. One might inquire whether there exists an integer $N > 0$ such that $c(\phi^N) = 0$ if the group $K = G \rtimes_\phi \mathbb{Z}$ is such that both K and G are word hyperbolic. One way to approach this question is to study splittings of K over $\mathbb{Z}$ induced by the outer automorphism ϕ of infinite order (use the theorem of Paulin [BS] to get a small action of K on an $\rtimes$-tree and the Rips machine [BF2] to promote this to a small action on a simplicial tree) and relate these to Dehn twists, Example 2.7 above. In effect, we are *conjecturing* that the word hyperbolic group $K = G \rtimes_\phi \mathbb{Z}$ is such that G is hyperbolic iff there exists $N > 0$ so that $c(\phi^N) = 0$, where the 'if' direction follows from Theorem B. If G is a word hyperbolic group and $\phi \in \mathrm{Aut}(G)$, does there exist $N > 0$ so that $c(\phi^N) = 0$?

Example 2.8. [4] Let G be the group of the presentation $\mathcal{P} = \langle a, b, b', c \mid [a, b] = c$, all other generators commute $\rangle$. Let ψ be the free group automorphism such that $b' \mapsto b'c$ and all other free generators are fixed. Then ψ induces the automorphism ϕ of G and we can form the semi-direct product $K = G \rtimes_\phi \mathbb{Z}$.

We claim that the inclusion $G < K$ distorts area and $c(\phi) > 0$. To see this note that G is the direct product of the 3-dimensional integral Heisenberg group H with $\mathbb{Z}$ and K is the 5-dimensional integral Heisenberg group. We proved in [Ge2] that the family of words $w_n = [b^n a^n b^{-n}, a^n]$ requires cubic filling area (in n) in H, and hence also in G (since H is a retract of G), but the family can be filled quadratically in K. It follows that the inclusion $G < K$ distorts area and hence $c(\phi) > 0$ by Theorem B.

We can relate the complexity $c(\phi)$ to group homology as follows. If $\phi \in \mathrm{Aut}(G)$ then by functoriality ϕ induces an automorphism $H_2(\phi)$ of $H_2(G, \mathbb{Z})$.

Proposition C. *If G is a finitely presented group and if $c(\phi) = 0$ for some $\phi \in \mathrm{Aut}(G)$, then $H_2(\phi)$ is of finite order in $\mathrm{Aut}(H_2(G, \mathbb{Z}))$.*

Proof. Since $c(\phi) = 0$ there is a finite presentation $\mathcal{P} = \langle x_1, x_2, \ldots, x_r \mid R_1, R_2, \ldots, R_s \rangle$ for G and an automorphism ψ of the free group F on the generators preserving the normal subgroup N of relations so that ψ induces ϕ and such that ψ permutes the cyclic words R_i up to inversion. Thus there exists $n > 0$ so that ψ^n fixes each cyclic word R_i, and hence there are $u_i \in F$ so that $\psi^n(R_i) = u_i R_i u_i^{-1}$ for $1 \le i \le s$.

By Hopf's formula [HS] there is a natural exact sequence

$$0 \to H_2(G, \mathbb{Z}) \to N/[F, N] \to H_1(F, \mathbb{Z}) \to H_1(G, \mathbb{Z}).$$

It follows that $H_2(\phi)$ is induced by the map induced by ψ on the quotient $N/[F, N]$ of N. But

$$\psi^n(R_i) = u_i R_i u_i^{-1} = [u_i, R_i] R_i \equiv R_i \pmod{[F, N]}.$$

Thus ψ^n induces the identity on $N/[F, N]$, whence $H_2(\phi^n) = (H_2(\phi))^n$ is the identity.

One can do the homology calculations explicitly for the group G in Example 2.8 and verify that $H_2(\phi)$ there is of infinite order, confirming the conclusion there that $c(\phi) \neq 0$.

[4]This example was also discussed in [Ge1].

Note also that an affirmative answer to the question raised immediately preceding Example 2.8 would imply that $H_2(\phi)$ was of finite order in $\mathrm{Aut}(H_2(G, \mathbb{Z}))$ for an automorphism ϕ of the hyperbolic group G.

Remark. Domingo Toledo pointed out to me that there is a connection between my question on the finiteness of the order of the automorphism $H_2(\phi)$, where ϕ an automorphism of the hyperbolic group G, and the question whether the Gromov pseudonorm on $H_2(G, \rtimes)$ is in fact a norm. If the latter assertion is true, then $H_2(\phi)$ is an isometry of the finite dimensional space $H_2(G, \rtimes)$ preserving the integral lattice $H_2(G, \mathbb{Z})$, and hence it is of finite order.

Remark. A recent paper of Rips and Sela shows that for certain hyperbolic groups G, including those which are torsion free and freely indecomposable, the subgroup of $\mathrm{Aut}(G)$ generated by Dehn twists and inner automorphisms is of finite index in $\mathrm{Aut}(G)$ [RS1]. Since Dehn twists and inner automorphisms induce the identity on H_2, it follows that $H_2(\phi)$ is of finite order for $\phi \in \mathrm{Aut}(G)$, where G is a hyperbolic group to which their results apply.

Next, there is a kind of suspension operation for converting length distortion into area distortion, which we illustrate in the next example (*cf.* also §8 below).

Example 2.9. Let G be the group of the presentation $\mathcal{P} = \langle x, y \mid yxy^{-1} = x^2 \rangle$ and let $H = \langle x \rangle < G$. The subgroup H is exponentially distorted in G (*cf.* [Gr2]) but there is no area distortion since H is free. However the inclusion $H \times \langle t \rangle < G \times \langle t \rangle$ distorts area, where $\langle t \rangle$ is an infinite cycle. To see this, consider the circuits $w_n = [x^{2^n}, t^{2^n}]$ whose area in the free abelian group $\langle x, t \rangle$ is 4^n for the standard presentation on these generators. However, by drawing the appropriate van Kampen diagram, one sees that the area of w_n in the presentation $\mathcal{Q} = \langle x, y, t \mid yxy^{-1} = x^2,\ [t, x],\ [t, y]\ \rangle$ for $G \times \langle t \rangle$ is at most $(2n + 3)2^n$, so there is distortion of area.

Remark. There are certainly examples of 1-relator presentations $\mathcal{P}$ and automorphisms ϕ of the group G defined by $\mathcal{P}$ so that ϕ is not $\mathcal{P}$-tame. Besides automorphisms of finite cyclic groups, it is shown that the figure 8 knot group has such an automorphism in [Ra] Theorem 1. However the last example is of finite order in $\mathrm{Out}(G)$ by Mostow rigidity, since the figure 8 knot complement is a complete hyperbolic manifold. I do not know any such example of infinite order in $\mathrm{Out}(G)$ for a 1-relator group G.

Example 2.10. Suppose X is a finite piecewise Euclidean simplicial complex with large links, so the universal cover $\widetilde{X}$ has a CAT(0)-metric [Gr1]. If $H \cong \mathbb{Z}^2 < \pi_1(X) = G$, then the inclusion $H < G$ does not distort area. The reason is that H stabilizes a flat plane Y in $\widetilde{X}$ by the theorem of Bridson-Gromov [Br][Gr1], and we can use nearest-point projection $\widetilde{X} \to Y$ to show that areas (defined in terms of number of faces of van Kampen diagrams) do not increase under the projection. This involves a certain amount of fussing, since the vertices of $\widetilde{X}^{(0)}$ do not project to vertices for an H-equivariant triangulation for Y, but one can budge them a bounded distance so that after budging they do.

Arguments like this one could be made more functorial if there were an intrinsic notion of "area" for certain length spaces, including at the very least the CAT(0)-spaces, but I do not yet see how to define such a notion in the absence of unique geodesic simplexes defined by a finite set of points.

3. Standard norms.

In this section we generalize the part of Theorem B which relates to preservation of weak area in a fibration over the circle. For this we need to recall from [Ge1] the notion of *standard norm* on a finitely generated module over a group ring.

Suppose the G is a group and M is a finitely generated $\mathbb{Z}G$ module. If we choose a finite set of generators, or, equivalently, a surjective homomorphism $\pi : C \to M$, where C is a finitely generated free based $\mathbb{Z}G$ module (so a free basis $\mathcal{B}$ for C is given), then we define the standard norm $|m|$ for $m \in M$ by the rule

$$|m| = \inf_{\substack{x \in C \\ \pi(x)=m}} |x|_1,$$

where $|x|_1$ is the ℓ_1-norm of x with respect to the basis $\{g \cdot b \mid g \in G,\ b \in \mathcal{B}\}$ for the underlying free abelian group of C. It is shown in [Ge1] Lemma 4.1 that the standard norm is well defined up to bi-Lipschitz equivalence for change of generators of M and that it is in fact a norm (and not merely a pseudonorm).

We need to recall a result from [Ge1]. Let $\mathcal{P}$ be a finite presentation for the group G with canonically associated 2-complex X. We have the chain complex $C_\star(\widetilde{X})$ for the universal cover $\widetilde{X}$, and the group of 1-cycles $Z_1(\widetilde{X})$ (which is the same as the group of cycles of the Cayley graph for $\mathcal{P}$) is finitely generated over $\mathbb{Z}G$, as follows from the surjective homomorphism $d_2 : C_2(\widetilde{X}) \to Z_1(\widetilde{X})$. Note that $C_2(\widetilde{X})$ is a free module over $\mathbb{Z}G$ with a basis given by choice of lifts of the 2-cells of X and associated ℓ_1-norm (which is independent of this choice of lifts). Thus we have the induced standard norm on $Z_1(\widetilde{X})$ for these generators.

We denote the weak area of a circuit w in the Cayley graph $\widetilde{X}^{(1)}$ of $\mathcal{P}$ by w-Area(w). Recall that this is the minimum area of an orientable surface diagram in $\widetilde{X}$ with its unique boundary component labelled w.

Theorem [Ge1]**.** *Suppose $\mathcal{P}$ is a finite presentation for the group G and w is a circuit in the Cayley graph of $\mathcal{P}$. Then we have*

$$\text{w-Area}(w) = |[w]|,$$

where $[w]$ is the class of w considered as a cycle in $Z_1(\widetilde{X})$.

Suppose now that ϕ is an automorphism of G which lifts to an automorphism ψ of the free group F on the generators of $\mathcal{P}$ preserving the normal subgroup N of relations. Then ψ induces an isomorphism $\psi_\star$ of the abelianization N_{ab}. Note

N_{ab} is identified with $Z_1(\widetilde{X}) = H_1(\widetilde{X}^{(1)}, \mathbb{Z})$, since N is identified with $\pi_1(X^{(1)})$, and hence the $\mathbb{Z}G$ module N_{ab} (the so-called relation module) is equipped with the standard norm.[5]

We can now state the result of this section.

Theorem C. *With the notations above, assume that* $\psi_\star$ *is an* isometry *for the standard norm on* N_{ab}. *Then the inclusion* $G < G \rtimes_\phi \mathbb{Z}$ *does not distort weak area.*

Proof. We refer to the part of the proof of Theorem B relating to weak area, where we perform surgery on the circuit $\mathbb{Z}$ separating the surface diagram S. Instead of invoking the Lemma there, we use the hypothesis that $\psi_\star$ is an isometry of N_{ab} for the standard norm and the interpretation above of standard norm as weak area to guarantee that the label u of $\mathbb{Z}$ has the same weak area as $\psi^{\pm 1}(u)$. So once again surgery on S can be done to reduce the number of faces. The remainder of the argument is unchanged, and Theorem C follows.

Remark. Example 2.8 also gives an example where the weak area is distorted, in passing from the product of the 3-dimensional integral Heisenberg group with $\mathbb{Z}$ to the 5-dimensional integral Heisenberg group.

Corollary. *Under the hypotheses of Theorem C, if the split extension* $G \rtimes_\phi \mathbb{Z}$ *is word hyperbolic, then* G *is also hyperbolic.* $\rtimes$

4. An example of distortion in automatic groups.

A finitely generated subgroup H of a word hyperbolic group G for which the inclusion $H < G$ does not distort lengths (in the word metrics) is necessarily quasiconvex and hence word hyperbolic; consequently by Proposition A of §2 there is no distortion of areas. In contrast, we give in this section an example of a finitely presented subgroup H of an automatic group G for which the inclusion $H < G$ does not distort lengths, yet area is distorted. The example is due to Martin Bridson [BBMS], who showed it to me in the context of a finitely presented subgroup H of an automatic group G which had an exponential Dehn function. My contribution is to note that the inclusion $H < G$ in Bridson's example does not distort lengths.

The starting point is to take a hyperbolic group K of the form $K = F \rtimes_\phi \langle t \rangle$, where t is an infinite cycle and where F is finitely generated free and nontrivial. Such groups exist and have the property that the automorphism ϕ of F is hyperbolic and hence has exponential growth; these facts can be found in [BF1]. Let $G = K \times L$, where L is free with free basis $\{a, b\}$. Then G is automatic, since the product of two automatic groups is automatic and hyperbolic groups are automatic [ECHLPT].

Let $H = \langle F, ta, tb \rangle < G$ and note that $H = F \rtimes \langle ta, tb \rangle$, where both ta and tb act on F by conjugation by the same monodromy ϕ (since a, b both centralize F). The fact that H has an exponential Dehn function[6] can be easily established by

[5] The correct transformation law for the ZG module structure on N_{ab} is given by $\psi_\star(g \cdot n) = \phi(g) \cdot \psi_\star(n)$ for $g \in G$ and $n \in N_{ab}$.

[6] The Dehn function of a finite presentation is its minimal isoperimetric function [Ge2].

various methods ([BGSS] §7, [BG], [Ge2]) and depends essentially on the exponential growth of ϕ. It follows that the inclusion $H < G$ distorts area since G satisfies the quadratic isoperimetric inequality [ECHLPT].

Note that killing the normal subgroup $F \rtimes G$ gives $\langle t\rangle \times L = \langle t\rangle \times (\langle a\rangle \star \langle b\rangle)$. Hence we distinguish three homomorphisms $G \to \mathbb{Z}$, denoted $\exp_t$, $\exp_a$, $\exp_b$ which correspond respectively to the exponent sums of the generators t, a, b in a word, where we use as generators for G a finite set of generators of F together with t, a, and b.

Lemma 4.1. *The kernel of the homomorphism $\chi = \exp_t - \exp_a - \exp_b$ is the subgroup H of G. It follows there is a short exact sequence of groups*

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\chi} \mathbb{Z} \longrightarrow 1.$$

Proof. Since $H = \langle F, ta, tb\rangle$, the algebraic number of times that t occurs in a word in the generators of H is the sum of the algebraic number of times that a and b occur in that word.

Conversely, assume a word w in the set of generators of G consisting of generators for F together with t, a, and b is such that $\exp_t(w) = \exp_a(w) + \exp_b(w)$. We may assume w has the form $ut^n v(a, b)$, where u is a word in the generators of F and v is a word in a, b. But then the word $w' = uv(ta, tb)$ represents the same element of G as w, and w' represents an element of H.

We now fix a set $\mathcal{A}$ of generators for F. We take the set $\mathcal{B} = \mathcal{A} \cup \{t, a, b\}$ as generators for G and take $\mathcal{C} = \mathcal{A} \cup \{ta, tb\}$ as generators for H. Lengths of words will be calculated with respect to these generators. Thus $|h|_{H,\mathcal{C}}, |g|_{G,\mathcal{B}}$ denote the word metric of $h \in H$, $g \in G$ in these generators.

Lemma 4.2. *We have $|h|_{H,\mathcal{C}} \leq 2|h|_{G,\mathcal{B}}$ for all $h \in H$.*

Proof. We may choose a geodesic word w for h in the metric for G of the form

$$w = u_1 \cdot t^{\epsilon_1} \cdot u_2 \cdot t^{\epsilon_2} \cdot \ldots \cdot u_r \cdot t^{\epsilon_r} \cdot v(a, b),$$

where each word u_i is in the generators $\mathcal{A}$ for F. This follows since all the a, b may be moved to the right in a geodesic word. By Lemma 4.1 we have $\sum_{i=1}^{r} \epsilon_i = \exp_a(v) + \exp_b(v)$. It follows from this and the centralizing properties of a, b that the word

$$w' = u_1 \cdot (ta)^{\epsilon_1} \cdot u_2 \cdot (ta)^{\epsilon_2} \cdot \ldots \cdot u_r \cdot (ta)^{\epsilon_r} \cdot (ta)^{-\sum \epsilon_i} \cdot v(ta, tb)$$

in the generators $\mathcal{C}$ for H represents h. Furthermore, the difference in lengths between w' and w is $|\sum \epsilon_i| \leq |h|_{G,\mathcal{B}}$. It follows that $|h|_{H,\mathcal{C}} \leq 2|h|_{G,\mathcal{B}}$, and the result is established.

It follows from Lemma 4.2 that the inclusion $H < G$ does not distort the word metrics.

Remark. The subgroup $F \times \langle a, b\rangle$ of G is a biautomatic subgroup of the biautomatic group G such that the inclusion homomorphism $F \times \langle a, b\rangle < G$ distorts area. In fact, the exponent of area distortion for the pair is 2, in the terminology of §9 below.

5. Upper bounds on area distortion.

Let $K = G \rtimes_\phi \mathbb{Z}$ where G is a finitely presented group and $\phi \in \mathrm{Aut}(G)$. It is easy to show using Britton's lemma [LS] that the distortion in the word metrics of G in K is at most exponential. We shall prove below, making use of Rapaport's theorem quoted in §2, that the distortion in area is at most an exponential of an isoperimetric function of a finite presentation for K.

Theorem 5.1. *Let G be a finitely presented group and let $K = G \rtimes_\phi \mathbb{Z}$ where $\phi \in \mathrm{Aut}(G)$. Let $\mathcal{P}$ be a finite presentation for G and let $\mathcal{Q}$ be a finite presentation for K containing $\mathcal{P}$ as a subpresentation. Then there is a constant $C > 1$ so that for each circuit w in the Cayley graph of $\mathcal{P}$ we have*

$$\mathrm{Area}_{\mathcal{P}}(w) \leq C^{\mathrm{Area}_{\mathcal{Q}}(w)}.$$

Proof. By Rapaport's theorem [Ra] we may choose the finite presentation $\mathcal{P}$ of G so that the automorphism ϕ is $\mathcal{P}$-tame. Let $\mathcal{P} = \langle x_1, x_2, \ldots, x_r \mid R_1, R_2, \ldots, R_s \rangle$ and let ψ be an automorphism of the free group on these generators preserving the normal subgroup of relations and so that ψ induces ϕ on G. We choose the presentation $\mathcal{Q}$ for K to be

$$\mathcal{Q} = \langle x_1, x_2, \ldots, x_r, t \mid R_i,\ t x_j t^{-1} = \psi(x_j);\ 1 \leq i \leq s,\ 1 \leq j \leq r \rangle.$$

Let D be a van Kampen diagram for w in $\mathcal{Q}$. Since there are no edges labelled t on the boundary, it follows that each edge labelled t is in a unique annulus in the interior of D obtained by stringing the HNN relators (*i.e.* those relators of $\mathcal{Q}$ involving t) together in a circuit. We call such an annulus a *t-ring*. Note that a crude upper bound on the number of t-rings is given by $\mathrm{Area}(D)$. We need the following

Lemma 5.2. *There is a constant $C_1 > 1$ depending only on $\mathcal{P}, \mathcal{Q}$, and ψ so that given a van Kampen diagram D for w with at least one t-ring, there is another van Kampen diagram D' for w so that D' has one fewer t-ring than D and so that $\mathrm{Area}(D') \leq C_1 \mathrm{Area}(D)$.*

Let us assume Lemma 5.2 for the moment and finish the proof of Theorem 5.1. If D is chosen originally as a *minimal* van Kampen diagram for w in $\mathcal{Q}$, then the number of t-rings in D is at most $\mathrm{Area}_{\mathcal{Q}}(w)$. We apply Lemma 5.2 successively to remove t-rings one at a time. After at most $\mathrm{Area}_{\mathcal{Q}}(w)$-steps we arrive at a van Kampen diagram for w in $\mathcal{P}$ with area at most $\mathrm{Area}_{\mathcal{Q}}(w) C_1{}^{\mathrm{Area}_{\mathcal{Q}}(w)}$. If we let $\sqrt{C} = \max(2, C_1)$, it follows that

$$\mathrm{Area}_{\mathcal{P}}(w) \leq C^{\mathrm{Area}_{\mathcal{Q}}(w)},$$

and the proof of the theorem is complete.

Proof of Lemma 5.2. Let $C_1 = \max_{1\leq i\leq s} \mathrm{Area}_{\mathcal{P}}(\psi^{\pm 1}(R_i))$. It follows that if u is in the normal subgroup of relations of $\mathcal{P}$, then $\mathrm{Area}_{\mathcal{P}}(\psi^{\pm 1}(u)) \leq C_1 \mathrm{Area}_{\mathcal{P}}(u)$.

Now consider an innermost t-ring A in the diagram D for w. Let the inner and outer boundary labels of A be u_0, u_1 respectively. These are elements of the free group on the generators of $\mathcal{P}$, and we have $u_1 = \psi^{\pm 1}(u_0)$, where the sign depends on the orientation of the t-edges of A. It follows that $\mathrm{Area}_{\mathcal{P}}(u_1) \leq C_1 \mathrm{Area}_{\mathcal{P}}(u_0)$. We now perform surgery on D, removing A and the interior diagram for u_0 and replacing them with a van Kampen diagram for u_1 in $\mathcal{P}$. The result is a new van Kampen diagram D' for w with one fewer t-ring and $\mathrm{Area}(D') \leq C_1 \mathrm{Area}(D)$. This completes the proof of the Lemma.

Corollary 5.3. *If $K = G \rtimes_\phi \mathbb{Z}$ where K is word hyperbolic and G is finitely presented, then G has an exponential isoperimetric function.* $\rtimes$

We shall now obtain an upper bound for the area distortion function in full generality in terms of isoperimetric functions by elementary methods.

Proposition 5.4. *Let $\mathcal{P}, \mathcal{Q}$ be finite presentations for groups H, G where $\mathcal{P}$ is a subpresentation of $\mathcal{Q}$ and $H < G$. Let f be the area distortion for the pair and let $\delta_{\mathcal{P}}$ be the Dehn function for $\mathcal{P}$. Then we have $f(n) \leq n\delta_{\mathcal{P}}(nM)$ for all $n \geq 0$, where M is the length of the longest relator of $\mathcal{Q}$.*

Furthermore, if H satisfies a polynomial isoperimetric inequality of degree d, then the area distortion of the pair (G, H) is bounded by a polynomial of degree d.

Proof. Let Δ, Γ be the Cayley graphs of $\mathcal{P}, \mathcal{Q}$ so Δ is a subgraph of Γ. Let w be an edge loop of Δ with $\mathrm{Area}_{\mathcal{Q}}(w) \leq n$.

Consider first the case where a minimal van Kampen diagram D for w in $\mathcal{Q}$ is a topological disc. In this case we have $\ell(w) \leq nM$ since nM is an upper bound on the number of edges of D. Thus $\mathrm{Area}_{\mathcal{P}}(w) \leq \delta_{\mathcal{P}}(nM)$ from the definition of Dehn function.

In the general case a minimal van Kampen diagram D for w in $\mathcal{Q}$ has maximal disc components $D_1, D_2, \ldots, D_k$ where each D_i is of area at most n and $k \leq n$. Let w_i be the boundary label of D_i and note that w_i is an edge circuit in Δ. From the special case above it follows that $\mathrm{Area}_{\mathcal{P}}(w_i) \leq \delta_{\mathcal{P}}(nM)$ and hence $\mathrm{Area}_{\mathcal{P}}(w) \leq n\delta_{\mathcal{P}}(nM)$. Thus $f(n) \leq n\delta_{\mathcal{P}}(nM)$.

Suppose now that H has a polynomial isoperimetric function of degree d. This means that if w is a circuit in the Cayley graph of $\mathcal{P}$ of length m, then $\mathrm{Area}_{\mathcal{P}}(w) \leq Am^d$ for $m \geq 1$, where $A > 0$ is a constant. In the notation of the preceding paragraph let $n_i = \mathrm{Area}_{\mathcal{Q}}(w_i)$. Since the length of w_i is at most Mn_i, we have $\mathrm{Area}_{\mathcal{P}}(w_i) \leq AM^d {n_i}^d$. Since $\sum_{i=1}^k n_i = n$, it follows that $\sum {n_i}^d \leq n^d$ and hence that $\mathrm{Area}_{\mathcal{P}}(w) \leq AM^d n^d$. It follows that $f(n) \leq AM^d n^d$ and the proof is complete.

Remark. Proposition 5.4 and its proof generalize Proposition A of §2. We shall be especially interested in applying it to get numerical invariants of area distortion of $\mathbb{Z}^n$-subgroups of finitely presented groups in §7 below.

Corollary 5.5. *If the finitely presented group G has a solvable word problem, then for every finitely presented subgroup $H < G$ the area distortion for the inclusion $H < G$ is a subrecursive function.*

Proof. Given a finite presentation for G and a finite set of generators for H, the word problem for H can be translated into that for G and solved. Since H is finitely presented, it follows that the Dehn function for each finite presentation of H is recursive. The result now follows from the Proposition.

Remark. I do not know an example of a finitely presented subgroup H of a finitely presented group G where the area distortion is greater than exponential. By Proposition 5.4 any isoperimetric function for H in such an example must grow faster than any exponential function.

6. Distortion with differing exponential growth rates.

In §2 we showed that $c(\phi) \neq 0$ is a necessary condition for area to be distorted under the inclusion $G < G \rtimes_\phi \mathbb{Z}$. However we do not know whether the condition is sufficient in general. In this section we prove a sufficient condition of homological character for the inclusion to distort area.

Let $\mathcal{P}$ be a finite presentation for the group G with canonically associated 2-complex X. The differential in the universal cover $\widetilde{X}$, $d_2 : C_2(\widetilde{X}) \to Z_1(\widetilde{X}) \subset C_1(\widetilde{X})$, is used to put the *standard norm* on the relation module $Z_1(\widetilde{X})$ (*cf.* §3 above) and the relation module projects canonically to $N/[F,N]$; here F is the free group on the generators of $\mathcal{P}$ and N is the normal subgroup of relations and we are using the fact that $Z_1(\widetilde{X}) = N/[N,N]$, which is basically Poincaré's formula ($\pi_1(\widetilde{X}^{(1)}) = N$ and $Z_1(\widetilde{X}) = H_1(\widetilde{X}^{(1)}) = H_1(N)$). Observe also that $N/[F,N]$ is a finitely generated abelian group, since it sits in the exact sequence (Hopf's formula [HS])

$$0 \to H_2(G,\mathbb{Z}) \to N/[F,N] \to H_1(F) \to H_1(G).$$

Thus $N/[F,N] \otimes \rtimes$ admits a unique equivalence class of (Minkowski) norms up to bi-Lipschitz equivalence, and these norms are compatible with the norm on the finite dimensional subspace $H_2(G, \rtimes)$.

Proposition 6.1. *There is a positive constant C so that for any based circuit w in $\widetilde{X}^{(1)}$ whose homotopy class is in the commutator subgroup $[F,F]$ (so w determines an element of $H_2(G)$ by Hopf's formula as above) we have*

$$|\rho(w)| \leq C \cdot \mathrm{Area}_{\mathcal{P}}(w);$$

here $\rho(w)$ is the class of w in $H_2(G, \rtimes)$ and the norm $|\rho(w)|$ is any (Minkowski) norm on the finite dimensional vector space $H_2(G, \rtimes)$.

Proof. The group $G = F/N$ acts trivially by conjugation on $N/[F,N]$ so the composition of maps $C_2(\widetilde{X}) \to Z_1(\widetilde{X}) \to N/[F,N]$ factors through $\mathbb{Z} \otimes_{\rtimes G} C_2(\widetilde{X}) = C_2(X)$. Thus each of these abelian groups is normed by taking the infimum of ℓ_1 norms of representatives in $C_2(\widetilde{X})$, and it follows that the norm induced on $N/[F,N] \otimes \rtimes$ is equivalent to any Minkowski norm. The statement about $H_2(G, ^{\rtimes})$ in the Proposition is a consequence of Hopf's formula and the fact mentioned above that a norm on $H_2(G, ^{\rtimes})$ is obtained by resticting the norm on $N/[F,N] \otimes^{\rtimes}$ to this subspace.

Proposition 6.2. *Suppose $\mathcal{P}$ is a finite presentation of the group G and ϕ is a $\mathcal{P}$-tame automorphism of G which is lifted by the free group automorphism ψ to the free group F on the generators preserving the normal subgroup N of relations of $\mathcal{P}$. Suppose that there are real numbers $C > 0$ and $\lambda > 1$ so that*

$$\ell(\phi^n(w)) \leq C\lambda^n \ell(w)$$

for all $n \geq 0$ and all $w \in F$. Suppose further that there exists $w \in N \cap [F,F]$ and real numbers $C' > 0$ and $\mu > 1$ so that

$$|H_2(\phi^n)(\rho(w))| \geq C'\mu^n$$

for all $n \geq 0$ (where ρ is as in the preceding result). If $\mu > \lambda$, then the inclusion $G < G \rtimes_\phi \mathbb{Z}$ distorts area.

Proof. Since $H_2(\phi^n)(\rho(w)) = \rho(\psi^n(w))$, it follows from the preceding result that $\text{Area}_{\mathcal{P}}(\psi^n(w)) \geq C'\mu^n$ for all $n \geq 0$.

We shall now construct a van Kampen diagram for $\psi^n(w)$ in $\mathcal{Q}$, a finite presentation for $G \rtimes_\phi \mathbb{Z}$. We choose $\mathcal{Q}$ to contain $\mathcal{P}$ as a subpresentation and to contain the stable letter t as additional generator and the additional relations $tx_it^{-1} = \psi(x_i)$, where the x_i are generators of $\mathcal{P}$. We start with a van Kampen diagram D for w in $\mathcal{P}$ and attach to D n t-rings to represent n iterated conjugations of w by the stable letter t. The result is a van Kampen diagram for $\psi^n(w)$ in $\mathcal{Q}$. If we use the hypothesis on the growth of the automorphism ψ and sum a geometric series, we see that there is a constant $C'' > 0$ so that $\text{Area}_{\mathcal{Q}}(\psi^n(w)) \leq C''\lambda^n\ell(w) + \text{Area}_{\mathcal{P}}(w)$ for all $n \geq 0$. Since $1 < \lambda < \mu$, we deduce that area is distorted under the inclusion $G < G \rtimes_\phi \mathbb{Z}$, and the proof is complete.

Example 6.3. Let $\phi \in \text{Gl}_4(\mathbb{Z})$ be given by $\alpha \oplus \alpha$, where $\alpha = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. We claim that area is distorted by the inclusion $\mathbb{Z}^4 < \mathbb{Z}^4 \rtimes_\phi \mathbb{Z}$.

To see this, note we can lift ϕ to an automorphism ψ of a free group F of rank 4 as the free product of lifts of two copies of α to free groups of rank 2. It follows that ψ is a geometric automorphism (*i.e.* it is induced by a homeomorphism of a compact bounded surface), and the Thurston theory of surface automorphisms implies that the growth of free words under iteration of ψ is governed by the dominant eigenvalue λ of α [FLP].

On the other hand, the dominant eigenvalue of $H_2(\phi) = \Lambda^2(\phi) : \Lambda^2(\mathbb{Z}^4) \to \Lambda^2(\mathbb{Z}^4)$ is λ^2. It follows that there is a word $w \in N \cap [F,F]$ so that the exponential growth rate of $\rho(w)$ under iteration by $H_2(\phi)$ is $\mu =: \lambda^2 > \lambda$. Thus the hypotheses of Proposition 6.2 are satisfied and there is distortion in area under the inclusion $\mathbb{Z}^4 \to \mathbb{Z}^4 \rtimes_\phi \mathbb{Z}$.

Remark. I do not yet know how to carry out an analog of the argument of Example 6.3 in general for PV automorphisms[7] ϕ of $\mathbb{Z}^n$, $n \geq 3$, so I cannot yet prove area distortion in $\mathbb{Z}^n < \mathbb{Z}^n \rtimes_\phi \mathbb{Z}$, as I expect to hold here.

In this connection, we shall discuss briefly the case of PV automorphisms in rank 3. Recall that if F is a finitely generated free group and if $\phi \in \mathrm{Out}(F)$, we say that ϕ is PV if $H_1(\phi) \in \mathrm{Aut}(H_1(F, \mathbb{Z}))$ is a PV automorphism. A PV outer automorphism ϕ is irreducible in the sense of [BH], as was proved in [GSt], so ϕ admits a minimal train track representative with expansion factor $\lambda > 1$. This implies that the exponential growth rate of the length of the all nontrivial cyclic words $\phi^n(w)$, $n \geq 0$, $1 \neq w \in F$, is λ (*i.e.* $\lim_{n\to+\infty} |\phi^n(w)|^{1/n} = \lambda$). Note that λ is at least as large as the absolute value of the dominant eigenvalue λ_1 of $H_1(\phi)$.

Hypothesis 6.4. If $\phi \in \mathrm{Out}(F)$ is PV and $\mathrm{rank}(F) = 3$, then the exponential growth rates λ, μ for ϕ, ϕ^{-1} respectively satisfy $\lambda \geq |\lambda_1| > \mu$, where λ_1 is the dominant eigenvalue of $H_1(\phi)$.

Example 6.5. Let $\phi : F(a, b, c) \to F(a, b, c)$ be given by $a \mapsto ac, b \mapsto a, c \mapsto b$. This automorphism is PV with $\lambda = \lambda_1 = 1.46\ldots.$ The inverse is given by $a \mapsto b, b \mapsto c, c \mapsto b^{-1}a$ and has exponential growth rate $\mu = 1.32\ldots.$ These statements hold since both maps are train track maps for the bouquet of circles, so the expansion rates can be calculated from the transition matrices for edges. The positivity of the map ϕ underlies the fact that $\lambda = \lambda_1$. I am grateful to M. Bestvina and M. Feighn for this example, which they calculated using Maple.[8]

It is perhaps surprising that the expansion rates for ϕ and ϕ^{-1} can differ, since this does not occur in the Thurston theory of surface homeomorphisms. We remark in this context that PV maps of free groups of rank at least 3 are never geometric [St] so this intuition fails.

Proposition 6.6. *If Hypothesis 6.4 is valid for the PV automorphism* $\phi \in \mathrm{Out}(F)$*, where* F *is free of rank 3, then the inclusion* $F_{ab} < F_{ab} \rtimes_{\phi_{ab}} \mathbb{Z}$ *distorts area; here* $F_{ab} = H_1(F, \mathbb{Z})$ *and* $\phi_{ab} = H_1(\phi)$*.*

Proof. We assume that $\det(\phi_{ab}) = 1$ and that the dominant eigenvalue $\lambda_1 > 1$ by replacing ϕ by ϕ^4 and adjusting the notation if necessary; statements about distortion of area are unaffected by this change. Let the eigenvalues of ϕ_{ab} be $\lambda_1, \lambda_2, \lambda_3$ with $\lambda_1 > 1$. Since $\lambda_1\lambda_2\lambda_3 = 1$ and since λ_2 and λ_3 are in the interior of the unit circle, it follows that $H_2(\phi_{ab}^{-1}) = \Lambda^2(\phi_{ab}^{-1})$ is PV with dominant eigenvalue $(\lambda_2\lambda_3)^{-1} = \lambda_1$. Thus the exponential growth rate of $H_2(\phi_{ab}^{-n})$ for $n > 0$ is $\lambda_1 \leq \lambda$, where λ is the exponential growth rate of the forward iterates of ϕ on free cyclic

[7]A monic polynomial in $\mathbb{Z}[x]$ is called PV if it has one eigenvalue counted with multiplicity in the exterior of the unit disc in $\rtimes$, and all other eigenvalues lie in the interior of the unit disc. The matrix ϕ is called PV if its characteristic polynomial is PV.

[8]M. Bestvina told me that for any given PV outer automorphism he can check by a combination of hand calculations and Maple whether Hypothesis 6.4 holds (and so far none tested has failed to satisfy it), but what is lacking is a technique for treating the question whether Hypothesis 6.4 is satisfied by *all* PV outer automorphisms.

words. On the other hand, the exponential growth rate of ϕ^{-n}, $n > 0$ on free words is $\mu < \lambda_1$, assuming Hypothesis 6.4 holds for ϕ. It follows from Proposition 6.2 that the inclusion $F_{ab} < F_{ab} \rtimes_{\phi_{ab}} \mathbb{Z}$ distorts area.

As an example, it follow that there is area distortion in the inclusion $F_{ab} < F_{ab} \rtimes_{\phi_{ab}} \mathbb{Z}$, where ϕ is the automorphism of Example 6.5.

Question. One might ask whether there is always area distortion in an inclusion $\mathbb{Z}^n < \mathbb{Z}^n \rtimes_\phi \mathbb{Z}$, where $n \geq 3$ and where ϕ is of infinite order in $\mathrm{Gl}_n(\mathbb{Z})$ (note that the area distortion is at most a quadratic polynomial, by Proposition 5.4). Related to this question and that preceding Example 2.8 is the following question: if G is a finitely presented group such that the inclusion $G < G \rtimes_\phi \mathbb{Z}$ does not distort area, does there exist $N > 0$ such that $c(\phi^N) = 0$?

7. Exponent of area distortion for torus bundles over the circle.

Definition 7.1. We consider pairs $(G, H), (G', H')$ of finitely generated groups where $H < G$ and $H' < G'$. A quasi-isometry[9] (f, f') of G with G' (where f maps G to G' and f' maps G' to G) is said to be adapted to the pairs $(G, H), (G', H')$ if $f(H)$ is contained in a Hausdorff neighborhood of H for the word metric of G' and $f'(H')$ is contained in a Hausdorff neighborhood of H for the word metric of G. Under these circumstances we write $(G, H) \sim (G', H')$. It is clear that these conditions are independent of choices of finite sets of generators for the groups.

A *geometric property* of a pair (G, H) is an invariant of all quasi-isometries adapted to pairs $(G, H), (G', H')$.

Examples.

7.1. The property of both G, H being finitely presented is a geometric property of pairs (G, H), as follows from conclusion (1) of Proposition B of §2. In addition, the length distortion function up to equivalence is a geometric property [Fa][Gr2].

7.2. If both G and H are finitely presented, the property of the inclusion $H < G$ not distorting area is a geometric property of pairs, as follows from conclusion (2) of Proposition B of §2. In addition, the area distortion function up to equivalence is a geometric property by §2 above.

It is especially interesting to look for geometric properties of pairs $(\mathbb{Z}^n \rtimes_\phi \mathbb{Z}, \mathbb{Z}^n)$. Since the area is undistorted if $n = 2$ and since we expect (although we cannot prove it yet) that the area is always distorted if $n \geq 3$ unless ϕ is of finite order in $\mathrm{Gl}_n(\mathbb{Z})$, we seek finer invariants.

Definition 7.3. Let f be the area distortion function for finite presentations $\mathcal{P} \subset \mathcal{Q}$ for $\mathbb{Z}^n < \mathbb{Z}^n \rtimes_\phi \mathbb{Z}$. The exponent $\alpha = \alpha(\phi)$ of area distortion is defined by

$$\alpha = \inf\{\beta \mid \lim_{n\to\infty} \frac{f(n)}{n^\beta} = 0\}.$$

[9] See [GH] for the notion of quasi-isometry of metric spaces and in particular for finitely generated groups.

Note that f is bounded above by a quadratic polynomial by Proposition 5.4 and one always has $f(n) \geq n$ (since $\mathrm{Area}_{\mathcal{Q}}(w) \leq \mathrm{Area}_{\mathcal{P}}(w)$ for all circuits w in the Cayley graph of $\mathcal{P}$), so we have $1 \leq \alpha \leq 2$. Note also that $\alpha = 1$ if the area is undistorted under the inclusion.

Proposition 7.4. *The exponent $\alpha(\phi)$ of area distortion is a geometric property of the pair $(\mathbb{Z}^n \rtimes_\phi \mathbb{Z}, \mathbb{Z}^n)$.*

Proof. This is immediate from the fact that the area distortion changes, under a quasi-isometry of pairs, by at most an affine change of coordinates in domain and range together with addition of a linear function. ⋊

Many results of the preceding sections can be interpreted in terms of the exponent of area distortion. We mention only one below in view of its potential interest.

Proposition 7.5. *Suppose that Hypothesis 6.4 holds for the PV automorphism $\phi \in \mathrm{Out}(F)$ where F is free of rank 3. Then $\alpha(\phi_{ab}) \geq \log_\mu |\lambda_1|$, where λ_1 is the dominant eigenvalue of ϕ_{ab} and where μ is the exponential growth rate of free words under iteration of ϕ^{-1}.*

Proof. Replace ϕ by ϕ^4 if necessary to assure that $\det(\phi_{ab}) = 1$ and $\lambda_1 > 1$. This does not change $\alpha(\phi_{ab})$ nor does it change $\log_\mu |\lambda_1|$. Now the exponential growth rate of $H_2(\phi^{-n})$ is λ_1, as was shown in the proof of Proposition 6.6; this is a lower bound for area in $\mathcal{P}$, the finite presentation for $\mathbb{Z}^3$. On the other hand the exponential growth rate of length of free words under iteration by ϕ^{-1} is μ. It follows that there is a nontrivial free word w which is a loop in the Cayley graph of $\mathcal{P}$ so that

$$\mathrm{Area}_{\mathcal{P}}(\phi^{-n}(w)) \geq C \cdot {\lambda_1}^n$$

but

$$\mathrm{Area}_{\mathcal{Q}}(\phi^{-n}(w)) \leq C' \cdot \mu^n;$$

here $n \geq 0$, $\mathcal{Q}$ is the finite presentation for $\mathbb{Z}^n \rtimes_\phi \mathbb{Z}$ and C, C' are positive constants.

If we let $a_n = \mathrm{Area}_{\mathcal{Q}}(\phi^{-n}(w))$, then it follows from these inequalities that $f(a_n) \geq C'' {a_n}^{\log_\mu \lambda_1}$, where f is the area distortion function and $C'' > 0$. Since $a_n \to \infty$ as $n \to +\infty$, it follows from the definitions that $\log_\mu \lambda_1 \leq \alpha(\phi_{ab})$, completing the proof.

Corollary 7.6. *Suppose that the Hypothesis 6.4 holds for the PV outer automorphism ϕ of the free group F of rank 3. Then $\mu \geq \sqrt{|\lambda_1|}$, in the notation of Proposition 7.5.*

Proof. This follows from the Proposition because $\alpha(\phi_{ab}) \leq 2$.

Remark. One can check that the automorphism in Example 6.5 has two complex nonreal eigenvalues. However there is only one quasi-isometry class of groups of the form $G = \mathbb{Z}^3 \rtimes_\phi \mathbb{Z}$, where $\phi \in \mathrm{Gl}_3(\mathbb{Z})$ is PV and has two complex nonreal eigenvalues. This follows since such a group G acts discretely and cocompactly by isometries for a suitable metric on $\rtimes^2 \times_{\mathbb{Z}}$ ([Wa] pp.123–124).[10]

In order to discuss this example a little further, we need the next result.

Proposition 7.7. *Suppose that $\mathcal{G}$ is a connected solvable Lie group with nilradical $\mathcal{N}$. If G, G' are lattices in $\mathcal{G}$, then $(G, G \cap \mathcal{N})$ and $(G', G' \cap \mathcal{N})$ are quasi-isometric by a quasi-isometry adapted to pairs.*

Proof. It follows from [R] Corollary 3.5 p. 50 that $G \cap \mathcal{N}$ is a lattice in the Lie group $\mathcal{N}$. If we extend the notion of quasi-isometry adapted to pairs (originally defined for finitely generated groups) to Lie groups in the obvious way, it follows that we have $(G, G \cap \mathcal{N}) \sim (\mathcal{G}, \mathcal{N}) \sim (G', G' \cap \mathcal{N})$, since all lattices in connected solvable Lie groups are uniform [R] Theorem 3.1.

Example 7.8. There is only one equivalence class of pairs (G, H) up to quasi-isometry adapted to pairs, where $G = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$, H is the normal $\mathbb{Z}^2$-subgroup, and A is a hyperbolic matrix in $\mathrm{Sl}_2(\mathbb{Z})$. To see this, note that G is a lattice in the 3-dimensional Lie group Sol; here $\mathrm{Sol} = \rtimes^2 \rtimes_\delta \rtimes$, where $\delta(t)(x, y) = (e^t x, e^{-t} y)$ [Sc]. Since the the subgroup $H < G$ is commensurable with a lattice in the nilradical $\rtimes^2$ of Sol, it follows that $(G, H) \sim (\mathrm{Sol}, \rtimes^2)$.

An open question in this connection is whether the pair (G, H) is *rigid*, *i.e.* whether every quasi-isometry of G is adapted to the pair (G, H).

Example 7.9. There is only one equivalence class of pairs (G, H) up to quasi-isometry adapted to pairs, where $G = \mathbb{Z}^3 \rtimes_\phi \mathbb{Z}$, H is the normal $\mathbb{Z}^3$ subgroup, and $\phi \in \mathrm{Gl}_3(\mathbb{Z})$ is PV and has two complex nonreal eigenvalues. To see this, note that G is a lattice in the solvable Lie group $\mathrm{Sol}_0^4 = \rtimes^3 \rtimes_\delta \rtimes$, where

$$\delta(t)(x, y, z) = (e^t x, e^t y, e^{-2t} z);$$

this follows from the discussion given in [Wa] p. 128. It follows from Proposition 7.7 above that H is commensurable with a lattice in the nilradical $\rtimes^3$ of Sol_0^4, and hence we have $(G, H) \sim (\mathrm{Sol}_0^4, \rtimes^3)$.

Proposition 7.10. *Suppose that $G = \mathbb{Z}^3 \rtimes_\phi \mathbb{Z}$, where $\phi \in \mathrm{Gl}_3(\mathbb{Z})$ is PV and has two complex nonreal eigenvalues. Then the inclusion of the normal $\mathbb{Z}^3$-subgroup in G distorts area.*

Proof. The PV automorphism arising from Example 6.5 has two complex nonreal eigenvalues and there is area distortion in including the $\mathbb{Z}^3$-normal subgroup in

[10] D. Toledo pointed out to me that the metric on $\mathbb{Z}^2 \times \rtimes$ for this isometric action is not the product Riemannian metric. Indeed, the product metric would be CAT(0) and the solvable group G cannot act cocompactly on a CAT(0) space. According to Toledo's calculations, the correct metric is

$$ds^2 = \frac{|dw|^2}{|\mathrm{Im}(w)|^2} + \mathrm{Im}(w)|dz|^2,$$

where w, z are complex parameters in the upper half plane and in $\rtimes$ respectively. He also noted that the manifold $G \backslash (\mathbb{Z}^2 \times \rtimes)$ is a complex surface which is not Kähler, known as the Inoue surface.

the split extension by Proposition 7.5. However by Example 7.9 there is only one quasi-isometry class of such pairs, so it follows that there is area distortion.

Remark. An element ϕ of $\mathrm{Gl}_3(\mathbb{Z})$ without an eigenvalue on the unit circle is either PV or its inverse is PV. I do not know yet whether there is area distortion in the split extension $\mathbb{Z}^3 < \mathbb{Z}^3 \rtimes_\phi \mathbb{Z}$ in the case when ϕ has 3 real eigenvalues, none of which lies on the unit circle in $_{\mathbb{Z}}$.

Furthermore, in the case of 2 complex eigenvalues, the exponent of area distortion is strictly greater than 1, as follows from Proposition 7.5 and Example 6.5. It would be interesting to know its exact value because of its connection with growth rates of free group automorphisms.

§8. Suspension theorem.

The purpose of this section is to establish the following result.

Theorem 8.1. *If $\phi \in \mathrm{Gl}_n(\mathbb{Z})$ is of infinite order, then the exponent of area distortion $\alpha(\phi \oplus 1)$ for the pair $(\mathbb{Z}^{n+1} \rtimes_{\phi\oplus 1} \mathbb{Z}, \mathbb{Z}^{n+1})$ satisfies the inequality $1 < \alpha(\phi \oplus 1) \leq 2$.*

Remark. This result implies that there is area distortion in the pair $(\mathbb{Z}^{n+1} \rtimes_{\phi\oplus 1} \mathbb{Z}, \mathbb{Z}^{n+1})$ in a strong sense. It is an open question whether $\alpha(\phi)$ itself is greater than 1 if $n \geq 3$ and ϕ is of infinite order.

The proof of Theorem 8.1 has two cases, depending on whether ϕ has an eigenvalue off the unit circle (in which case it has an eigenvalue of absolute value greater than 1) or whether ϕ has all eigenvalues on the unit circle, in which case some power of ϕ is unipotent (this follows from the well-known fact that if all roots of a monic integral polynomial are on the unit circle, then these roots are roots of unity).

We denote by $\|\phi\|$ the sup norm of the entries of the matrix ϕ. We shall also consider the sup norm $\|v\|$ of vectors $v \in \mathbb{Z}^n$. Observe that the sup norm is bi-Lipschitz equivalent to the ℓ_1-norm.

We define the growth function $g : \rtimes \to \rtimes$ of the matrix ϕ by $g(k) = \|\phi^k\|$, which we consider up to the equivalence relation given by affine change of coordinates in domain and range with positive scale factors in these affine functions. We say that the growth of ϕ is exponential (resp. polynomial of degree d) if g is equivalent to the function $k \mapsto e^k$ (resp. $k \mapsto k^d$). Let the standard basis for the column space $\mathbb{Z}^n$ be denoted by $e_1, e_2, \ldots, e_n$.

Lemma 8.2. *The growth function g is either exponential or polynomial. It is polynomial iff all eigenvalues of ϕ lie on the unit circle, in which case some positive power of ϕ is unipotent. If it is exponential, then the exponential rate of growth is $|\lambda|$, where λ is an eigenvalue of ϕ of largest absolute value.*

This is proved in [BG] Theorem 2.1.

Lemma 8.3. *Suppose that the growth function g of ϕ is exponential. Let λ be an eigenvalue of ϕ which is of largest absolute value $|\lambda| > 1$. Then there is a number*

k, $1 \leq k \leq n$, and an infinite sequence $n_i \to \infty$ as $i \to \infty$ so that

$$\|\phi^{n_i}\| = \|\phi^{n_i}(e_k)\|$$

for all numbers $i \geq 1$.

Proof. The growth function g of the matrix ϕ is equivalent to $i \mapsto e^i$ so the largest entries of the matrices ϕ^i increase exponentially at the exponential rate $|\lambda|$. Since there are only n-columns in the matrix, this maximum must be attained infinitely often in one of the columns, say the k-th column. It follows that

$$\|\phi^{n_i}\| = \|\phi^{n_i}(e_k)\|$$

for infinitely many n_i, and the result is established.

It follows from the preceding result that there are positive numbers C, C' so that

$$C'|\lambda^{n_i}| \leq |\phi^{n_i}(e_k)|_1 \leq C|\lambda^{n_i}|$$

for all numbers i in the case of exponential growth, where $|v|_1$ is the ℓ_1-norm of the vector v.

We can now give the proof of Theorem 8.1 in case the growth function g of ϕ is exponential. We follow the notation of Lemma 8.2 and the preceding paragraph. Let $\mathcal{P}$, $\mathcal{Q}$ be finite presentations for $\mathbb{Z}^{n+1}, \mathbb{Z}^{n+1} \rtimes_{\phi\oplus 1} \mathbb{Z}$ with $\mathcal{P}$ a subpresentation of $\mathcal{Q}$ and using the obvious generators $e_i,\ 1 \leq i \leq n+1$, for $\mathbb{Z}^{n+1}$ and an additional stable letter t for the HNN-extension which is here the semi-direct product. In defining $\mathcal{Q}$, we need to lift $\phi \oplus 1$ to a free group automorphism ψ of the free group on $n+1$ generators. We do this in such a way that ψ is the free product of a lift ψ' of ϕ to the free group of rank n with the identity on the last free basis element, $\psi = \psi' \star 1$. Observe that there is a positive constant M so that for all free words w we have $\ell(\psi^k(w)) \leq M^k \ell(w)$. Let $L_{n_i} = |\phi^{n_i}(e_k)|_1$.

We consider the free words $w_i = [\psi^{n_i}(e_k), e_{n+1}^{M^{n_i}}]$ and observe that it follows from an H_2-lower bound estimate of area in a free abelian group (*cf.* §6 above) that

$$\text{Area}_{\mathcal{P}}(w_i) \geq L_{n_i} M^{n_i} \geq C'|\lambda^{n_i}| M^{n_i}. \tag{8.4}$$

On the other hand, we can find a van Kampen diagram for w_i in $\mathcal{Q}$ which is obtained from a filling of the free word $[e_k, e_{n+1}^{M^{n_i}}]$ in $\mathcal{P}$ by adjoining n_i t-rings corresponding to n_i successive conjugations by the stable letter t. We can estimate the area of this diagram by summing a geometric series, and the result is that

$$\begin{aligned} \text{Area}_{\mathcal{Q}}(w_i) &\leq C_1(|\lambda|^{n_i+1} + (2n_i+1)M^{n_i}) \\ &\leq C_2(|\lambda|^{n_i} + (2n_i+1)M^{n_i}, \end{aligned} \tag{8.5}$$

where C_1, C_2 are positive constants. We may assume without loss of generality that $M > |\lambda|$ by modifying our initial choice of M. It follows from (8.4) and (8.5) and the definition of the exponent of area distortion that $\alpha(\phi\oplus 1)\geq \big(1+\dfrac{\log|\lambda|}{\log M}\big)>1$. This completes the proof of Theorem 8.1 in case the growth function of ϕ is exponential.

We proceed now to the case where ϕ has polynomial growth. In this case there exists $N>0$ so that ϕ^N is unipotent. Since the exponent of area distortion is not changed by replacing ϕ by ϕ^N, we may assume without loss of generality that ϕ is unipotent. In addition, $\alpha(\phi)\leq\alpha(\phi\oplus\psi)$, so it suffices to consider the case where ϕ is a fundamental n-by-n Jordan block for eigenvalue 1 with $n\geq 2$ (since ϕ is given of infinite order), which we denote by $J_n(1)$. This last step makes use of the fact that if two matrices $\phi,\phi'\in \mathrm{Gl}_n(\mathbb{Z})$ have the same Jordan canonical forms, then the semi-direct products G,G' for these monodromies are commensurable and the corresponding pairs $(G,\mathbb{Z}^n),(G',\mathbb{Z}^n)$ are quasi-isometric by a quasi-isometry adapted to the pairs (*cf.* [BG] Proposition 5.5).

The matrix $J_n(1)$ corresponds to the linear transformation given in terms of the standard basis for $\mathbb{Z}^n$ by $e_i\mapsto e_i+e_{i+1},\ 1\leq i\leq n-1,\ e_n\mapsto e_n$. In this case, the subgroup S of $\mathbb{Z}^n$ spanned by e_{n-1},e_n is invariant under ϕ and the matrix of the restriction of ϕ to this subgroup is $J_2(1)=\begin{pmatrix}1&0\\1&1\end{pmatrix}$. Since the area function of the subgroup S is undistorted in including S in $\mathbb{Z}^n$ (there is a retraction for the inclusion), it will suffice to prove that $\alpha(\phi\oplus 1)>1$ when $\phi=J_2(1)$ (in effect, the area distortion only increases by inclusion in the bigger semi-direct product).

We are thus reduced to showing that $\alpha(\phi\oplus 1)>1$ when $\phi=J_2(1)$. Choose the standard basis e_1,e_2 for Z^2 so ϕ is given by $e_1\mapsto e_1+e_2,\ e_2\mapsto e_2$. Lift $\phi\oplus 1$ to the free group on the free basis e_1,e_2,e_3 to get the automorphism ψ given by the formula $e_1\mapsto e_1e_2, e_2\mapsto e_2, e_3\mapsto e_3$. Choose the presentations $\mathcal{P}$ and $\mathcal{Q}$ as before, so that the stable letter t for the HNN-extension $\mathbb{Z}^3\rtimes_{\phi\oplus 1}\mathbb{Z}$ conjugates the free generators e_i by the automorphism ψ.

Consider the free words $w_n=[\psi^n(e_1^n),e_3^{n^2}]=[(e_1{e_2}^n)^n,e_3^{n^2}]$. One sees that $\mathrm{Area}_{\mathcal{P}}(w_n)=n^4+n^3$. On the other hand, $w_n=\psi^n([e_1^n,e_3^{n^2}])$, so we can construct a van Kampen diagram for w_n in $\mathcal{Q}$ by taking a diagram for $[e_1^n,e_3^{n^2}]$ in $\mathcal{P}$ and adjoining n t-rings corresponding to n successive conjugations by the stable letter t. An easy calculation shows that $\mathrm{Area}_{\mathcal{Q}}(w_n)\leq 3n^3+2n(1+2+3+\cdots+(n-1))\leq 4n^3$. It follows from these calculations that the exponent of area distortion is at least 4/3.

Thus the exponent of area distortion $\alpha(\phi\oplus 1)$ for the case where ϕ is of polynomial growth and of infinite order is strictly greater than 1, and the proof of Theorem 8.1 is complete.

We shall now give examples $\phi\in\mathrm{Gl}_n(\mathbb{Z})$ where $\alpha(\phi\oplus 1)=2$.

We recall first some aspects of the Frobenius-Perron theory of positive matrices (*cf.* [Se]). If A is a square matrix with real entries, we write $A\geq 0$ if each entry of A is nonnegative and we write $A>0$ if each entry of A is positive. If $A\geq 0$, we say that A is *primitive* if $A^k>0$ for some $k>0$. Recall that if $A\geq 0$ is primitive, then there exists a positive dominant eigenvalue λ of A which is a simple root of the characteristic equation of A. In addition, if v is any positive column vector, the iterates $A^k v,\ \ k > 0$, grow like $\lambda^k |v|_1$.

Theorem 8.6. *If $\phi \in \mathrm{Gl}_n(\mathbb{Z})$ is such that*

(1) *the matrix ϕ is a finite product of elementary transvections of the form $E_{ij}(a)$ for positive integers a (and variable i, j),*
(2) *the matrix ϕ is primitive, and*
(3) *$\lambda > 1$ for the dominant eigenvalue λ of ϕ,*

then $\alpha(\phi \oplus 1) = 2$.

Proof. The factorization of ϕ as a product of positive elementary transvections enables us to lift ϕ to $\phi_\star$, an automorphism of the free group F_n of rank n, by Dehn twists on free basis elements involving only positive twists. If we take a positive word u in the free group (so u involves only elements of the free basis and no inverses), it follows from the remarks preceding the statement of the Theorem that the forward iterates $\phi_\star^k(u)$, $k > 0$, grow like a constant multiple of λ^k.

Now look at $\phi \oplus 1$ and lift it to $F_{n+1} = F_n \star \mathbb{Z}$ by $\phi_\star \star 1$. Let e_{n+1} be the last free basis element of F_{n+1} in the free product decomposition above and consider the free words $w_k = [\phi_\star^k(u), {e_{n+1}}^{L_k}]$, where L_k is the length of the (positive) word $\phi_\star^k(u)$. It is clear that the area of w_k in a presentation for $\mathbb{Z}^n$ is $L_k^2 \sim \lambda^{2k}$. On the other hand, a van Kampen diagram can be constructed for w_k in a presentation for $\mathbb{Z}^{n+1} \rtimes_{\phi\oplus 1} \mathbb{Z}$ with area at most of the order of magnitude $\lambda^{k+1} + k\lambda^k$. It follows from the definition that $\alpha(\phi \oplus 1) \geq 2$. Proposition 5.4 shows the opposite inequality holds, and the proof of the Theorem is complete.

Remark. It should be possible to weaken the hypotheses significantly in the preceding Theorem to get the same conclusion.

§9. An example of fractional exponent of area distortion.

When H is a finitely presented subgroup of the finitely presented group G such that H has a polynomial isoperimetric function of degree d, then the exponent of area distortion $\alpha = \alpha(G, H)$ can be defined by Definition 7.3 as

$$\alpha = \inf\{\beta \mid \lim_{n\to\infty} \frac{f(n)}{n^\beta} = 0\};$$

here f is the area distortion function defined for finite presentations of G, H. From Proposition 5.4 it follows that one has $1 \leq \alpha \leq d$. In this section we shall give an example where α lies in the open interval $(1, d)$. In our example both G and H are automatic groups, so $d = 2$.

Let $H = F(x, y) \times \langle z \rangle$, where $F(x, y)$ is free with basis x, y and where z is of infinite order. Let G be the mapping torus of the automorphism ϕ of H given by $x \mapsto xy,\ y \mapsto y,\ z \mapsto z$, so $G \cong H \rtimes_\phi \langle t \rangle$, where t is of infinite order. Note also that z is central in G and $F(x, y)$ is stabilized by ϕ, so $G = (F(x, y) \rtimes_\phi \langle t \rangle) \times \langle z \rangle$. In fact both G and H are automatic groups. This is clear for H, since finitely generated free groups are automatic and the direct product of two automatic groups is automatic. There are several ways of see that $F(x, y) \rtimes_\phi \mathbb{Z}$ is automatic; one way is to observe that it is the fundamental group of a compact Haken manifold with nonempty boundary and to quote results of [ECHLPT]. It follows from the product theorem that G is also automatic. Our main result is the following

Theorem 9.1. *The exponent of area distortion α for the pair (G, H) above satisfies $4/3 \leq \alpha \leq 3/2$.*

Remark. The upper bound is the more difficult result, while the lower bound will be established by explicitly constructing a sequence of words and estimating their areas. I suspect that the lower bound 4/3 is the actual value of α in this example, since I have not been able to find a sequence of words which distorts the area more, but my analytic tools are not sufficiently refined yet to get the best upper bound.

We begin the proof by fixing finite presentations $\mathcal{P} \subset \mathcal{Q}$ for $H < G$. Let $\mathcal{P} = \langle x, y, z \mid [x, z],\ [y, z]\rangle$ and let $\mathcal{Q} = \langle x, y, z, t \mid [x, z],\ [y, z],\ x^t = xy,\ y^t = y\rangle$. Consider the free words $w_n = [(xy^n)^n, z^{n^2}]$ which represent loops in the Cayley graph of $\mathcal{P}$. We have $\mathrm{Area}_{\mathcal{P}}(w_n) = n^4 + n^3$, while the argument in the next to the last paragraph of the proof of Theorem 8.1 shows that $\mathrm{Area}_{\mathcal{Q}}(w_n) \leq 4n^3$. Note that in that latter argument, one never used the commutativity of x with y (e_1 and e_2 in the earlier notation) so the diagram constructed there is a van Kampen diagram for w_n in the present situation. It follows as earlier that $\alpha \geq 4/3$.

The second inequality in the Theorem depends on an analysis of a minimal van Kampen diagram D in $\mathcal{Q}$ of a loop w in the Cayley graph of $\mathcal{P}$. We may assume that D is a topological disc, for the general case can be reduced to this case. The description $G = (F(x, y) \rtimes_\phi \langle t\rangle) \times \langle z\rangle$ shows there is a homomorphism $G \to \mathbb{Z}$ taking x, y, t to the identity and taking z to a generator of $\mathbb{Z}$. This homomorphism corresponds to a map $K(\mathcal{Q}) \to S^1$, where $K(\mathcal{Q})$ is the 2-complex canonically associated to the presentation $\mathcal{Q}$. We make this map transverse to a general point p of S^1 and take the preimage P of p under the composition of maps $D \to K(\mathcal{Q}) \to S^1$. The set P is a 1-manifold properly imbedded in D. As usual we may assume there are no circles in P, for otherwise we could surger D and reduce the area without changing the boundary label.

Each arc component I_i of P is incident to a "z-corridor" C_i, in the language of [BG]. Namely, C_i consists of all 1- and 2-cells of D which meet I_i. Each of these 1-cells is labelled z and the portion of ∂C_i not labelled z consists of two arcs each labelled u_i, where u_i is a word in x, y, t and their inverses. Note that the area of C_i is the length $\ell(u_i) =: L_i$ of the word u_i. This follows from the fact that the relators involving the letter z are just commutators with the other generators of the presentation $\mathcal{Q}$.

Now each word u_i represents an element of $F(x, y)$. This is a consequence of the fact that the corridor C_i separates D, so the z-letters on each side on the boundary cancel in pairs. Let v_i be the reduced word in $F(x, y)$ representing u_i. Observe that v_i and u_i together form a loop in the Cayley graph of $\mathcal{Q}$, so we let $A_i = \mathrm{Area}_{\mathcal{Q}}(v_i \cdot {u_i}^{-1})$. Observe that a minimal van Kampen diagram for the loop $v_i \cdot {u_i}^{-1}$ will contain no z-edges at all, since these would form z-rings in the interior which could be surgered out.

The next result is the key technical tool in the argument.

Lemma 9.2. *Let $u = u(x, y, t)$ be a word in the generators of $\mathcal{Q}$ which represents the reduced word v in $F(x, y)$. Then we have*

$$\ell(v) \leq C(L + L^{1/2}A^{1/2});$$

here $L = \ell(u)$, $A = \mathrm{Area}_{\mathcal{Q}}(u \cdot v^{-1})$, and C is a constant depending only on $\mathcal{Q}$ but not on u, v.

Remark. Note that Lemma 9.2 is a statement about relation of lengths and areas for the inclusion of the subgroup $F(x, y)$ in the group $F(x, y) \rtimes_\phi \mathbb{Z}$. Geometrically, this is the inclusion of the fibre in a bundle over the circle, where the fibre is a punctured 2-torus and the monodromy is a parabolic homeomorphism of the fibre.

We postpone the proof of Lemma 9.2 until later. Assuming the Lemma, we shall complete the proof of Theorem 9.2. Note that the geodesic words v_i in $F(x, y)$ enable us to construct a van Kampen diagram D' for w in $\mathcal{P}$, as follows. The corridor C_i of D is replaced by a corridor C_i' of D' expressing the commutativity of z with v_i. Each loop constructed of arcs v_i and arcs of w not involving z is a loop in the Cayley graph for $F(x, y)$, a tree, so has area 0 and hence can be filled without any faces in $\mathcal{P}$, completing the description of D'. Thus we have $\mathrm{Area}_{\mathcal{P}}(w) \leq \mathrm{Area}(D') \leq \sum_i \mathrm{Area}(C_i') \leq \sum_i \ell(v_i)$. We next use Lemma 9.2 to estimate $\ell(v_i)$ to get $\mathrm{Area}_{\mathcal{P}}(w) \leq C \sum_i (L_i + {L_i}^{1/2} {A_i}^{1/2})$.

Observe that $\sum_i L_i = \sum_i \mathrm{Area}(C_i) \leq \mathrm{Area}_{\mathcal{Q}}(w)$. Also note that $A_i \leq \mathrm{Area}_{\mathcal{Q}}(w)$. The reason for the last assertion is that if we take one of the two connected components X of $D - C_i$ and remove all the remaining z-corridors from X, gluing together their boundaries, and fold the edges on the boundary word if backtracks arise, what we get is a van Kampen diagram ${D_i}''$ for the word $v_i \cdot u_i^{-1}$. Thus we have $A_i \leq \mathrm{Area}({D_i}'') \leq \mathrm{Area}(D) = \mathrm{Area}_{\mathcal{Q}}(w)$. If we put this information into the last result of the preceding paragraph, we obtain $\mathrm{Area}_{\mathcal{P}}(w) \leq C(\mathrm{Area}_{\mathcal{Q}}(w) + \mathrm{Area}_{\mathcal{Q}}(w)^{1/2} \sum_i {L_i}^{1/2})$. Next we use the Cauchy-Schwartz inequality to estimate $\sum_i {L_i}^{1/2} \cdot 1 \leq (\sum_i L_i)^{1/2} (\sum_i 1)^{1/2} \leq (\mathrm{Area}_{\mathcal{Q}}(w))^{1/2} (\mathrm{Area}_{\mathcal{Q}}(w))^{1/2} = \mathrm{Area}_{\mathcal{Q}}(w)$. Putting this into the previous estimate, we get

$$\begin{aligned} \mathrm{Area}_{\mathcal{P}}(w) &\leq C(\mathrm{Area}_{\mathcal{Q}}(w) + \mathrm{Area}_{\mathcal{Q}}(w)^{3/2}) \\ &\leq C' \mathrm{Area}_{\mathcal{Q}}(w)^{3/2}, \end{aligned}$$

where C' is a constant. It follows from this last inequality and the definition of exponent of area distortion that $\alpha \leq 3/2$. Thus Lemma 9.2 implies Theorem 9.1.

We now proceed to demonstrate Lemma 9.2.

We begin by partitioning the reduced word v in $F(x, y)$ into disjoint subwords v_j such that $v = v_1 v_2 \ldots v_s$ without cancellation according to the following rules.

If we encounter a subword xy^kx of v, then xy^k is one of the subwords v_j. If we encounter the subword $x^{-1}y^kx^{-1}$ of v, then y^kx^{-1} is one of the v_j. If we encounter the subword xy^kx^{-1} of v, then xy^kx^{-1} itself is one of the v_j. And finally, if we encounter the subword $x^{-1}y^kx$ of v, then y^k is one of the v_j. The ordering on the v_j is in order of appearance as we read v from left to right.

Example. If $v = (yx^{-1})(y^{-4})(xy^2)(xy^{-2}x^{-1})$, then $v_1 = yx^{-1},\ v_2 = y^{-4},\ v_3 = xy^2,\ v_4 = xy^{-2}x^{-1}$.

The reason for this partition is the action of the automorphism ϕ on v. We have namely

(1) $\dots xy^kx \dots \mapsto \dots xy^{k+1}x \dots$,
(2) $\dots x^{-1}y^kx^{-1} \dots \mapsto \dots x^{-1}y^{k-1}x^{-1} \dots$,
(3) $\dots xy^kx^{-1} \dots \mapsto \dots xy^kx^{-1} \dots$, and
(4) $\dots x^{-1}y^kx \dots \mapsto \dots x^{-1}y^kx \dots$.

This calculation shows that a subword v_j arising in situations (3) and (4) does not change under application of ϕ, whereas in each of situations (1) and (2) the number of y's changes by 1. The number of factors in the partition is also invariant under application of ϕ, so the effect of ϕ can be calculated by applying it to each of the factors v_j in order, then taking the product of the results. No cancellation arises in the process.

Consider now a minimal van Kampen diagram D for the word $u \cdot v^{-1}$ in the statement of Lemma 9.2. We have already observed that there are no z-rings. But note that the group $F(x,y) \rtimes_\phi \langle t \rangle$ has the presentation $\langle x, y, t \mid [t,y],\ x^{-1}tx = yt \rangle$, which makes it evident that this group is an HNN extension with base group the $\mathbb{Z}^2$-subgroup $\langle t, y \rangle$ and stable letter x and associated infinite cyclic subgroups $\langle t \rangle$ and $\langle yt \rangle$. It follows from this that D contains no x-rings, for otherwise we could surger them out, thereby reducing the area without changing the boundary label. Corresponding to each occurrence of the letter x in the subword v of the boundary of D is an x-corridor running from the v-side to the u side of the boundary. The reason is that no nontrivial word of the form t^n or $(yt)^n$ represents an element of $F(x,y)$, as would be the case if an x-corridor connected two occurrences of x on the v-side of ∂D.

On the other hand, an x-corridor can connect two occurrences of x on the u-side of ∂D. In this case, we remove the outermost such x-corridors relative to the u-side and everything they contain between them and the u-side. This changes the u-side of the diagram, but does not change the v-side. The area is reduced in the surgery, but the length of the u-side increases at most by a factor of 2. The reason for the last assertion is that the subgroup $\langle t \rangle$ is undistorted in length in $F(x,y) \rtimes_\phi \langle t \rangle$ whereas lengths in the subgroup $\langle yt \rangle$ can be reduced in the ambient group by a factor of at most 2, in the word metric for the given generators.

After these surgeries, we have a van Kampen diagram D' for $u' \cdot v^{-1}$, where $\ell(u') \leq 2\ell(u)$, where $\mathrm{Area}(D') \leq \mathrm{Area}(D)$, and where every x-corridor connects an x on the v-side of the boundary to an x on the u'-side.

We now bring in the partition of $v = v_1v_2 \dots v_s$ above. Corresponding to this is a partition of the word $u' = u_1u_2 \dots u_s$, where u_j involves as many x's as v_j (in any case, at most 2 x's) and where u_j is read off from the u'-side of the boundary. The x's in u_j are at the beginning or end, or, in case (4) above, there are no x's. The rest of u_j is a word ${u'}_j$ in y and t and their inverses. Also let v'_j be the largest subword of v_j involving no x's. Now the words ${u'}_j$ and ${v'}_j$ represent subwords of $\partial D'$ between two successive x-corridors. Let A_j denote the area of this subdiagram of D'.

Lemma 9.3. *In the notation above we have $\ell(v'_j) \leq C(\ell(u'_j) + {A_j}^{1/2})$ for each index j.*

Let us assume Lemma 9.3 for the moment and finish the proof of Lemma 9.2. We may assume that $C \geq 1$ by increasing it if necessary. Then it follows from Lemma 9.3 that $\ell(v_j) \leq C(\ell(u_j) + {A_j}^{1/2})$. Thus we have

$$\begin{aligned}\ell(v) = \sum_j \ell(v_j) &\leq C\sum_j(\ell(u_j) + {A_j}^{1/2}) \leq 2CL + C\sum_j {A_j}^{1/2}\\ &\leq 2CL + C(\sum_j A_j)^{1/2}(\sum_j 1)^{1/2} \text{ (using Cauchy-Schwartz)}\\ &\leq 2CL + CA^{1/2}L^{1/2} \leq C'(L + A^{1/2}L^{1/2}),\end{aligned}$$

where C' is a constant. This completes the proof that Lemma 9.3 implies Lemma 9.2.

Proof of Lemma 9.3. There are 4 main cases to consider, corresponding to the four situations in the definition of the partition of the word v. Cases (3) and (4) are especially easy, once one has seen how to handle the difficult cases (1) and (2). Cases (1) and (2) are symmetric, so we consider case (1).

So we assume that v_j is defined by an occurrence of the subword xy^kx in v and u_j is the analogous portion of the word u' and v'_j, u'_j are the corresponding words with the x's left off. These words on $\partial D'$ are between two x-corridors with the x-letters oriented in the same direction and A_j is the area of the region R_j between these two x-corridors. It follows that the two remaining sides of R_j are labelled t^m and $(yt)^n$ for suitable integers m, n and the region R_j is a van Kampen diagram in the presentation for $\mathbb{Z}^2$ given by $\langle t, y \mid [t, y]\rangle$. Hence we can map R_j to the Euclidean plane with the square lattice; this map does not increase distances nor areas. Furthermore, if the t-axis in the plane is vertical and the y-axis is horizontal, the segment of ∂R_j labelled t^m maps to a vertical line segment and the segment labelled $(yt)^m$ maps to a staircase approximating a line of slope 1. The segment v'_j maps to a horizontal line segment, whereas the segment u'_j maps to some path on the square lattice making a closed loop with the other paths. Then Lemma 9.3 in case (1) follows from the following isoperimetric inequality for the Euclidean plane.

Lemma 9.4. *Suppose we draw a vertical line segment S from the point p on the horizontal axis ending at the point p' and we draw a line segment T of slope 1* *beginning at the point* q *on the horizontal axis and ending at the point* q'. *Let* u *be any rectifiable path from* p' *to* q'. *Then* $d(p,q) \leq C(\ell(u) + A^{1/2})$. *Here* C *is a constant,* $d(p,q)$ *is the Euclidean distance,* $\ell(u)$ *is the length of the path* u, *and* A *is the minimal area of a genus 0 surface spanned by the closed curve* $\mathfrak{c}$ *comprised of the paths* $[p,q]$, T, u, *and* S.

Proof. There are several cases to consider depending on the position of S relative to T. We carry out the argument in detail for one of the cases, when S and T are disjoint and T is entirely to the left of S and both S and T are in the upper half plane. The remaining cases will be left to the reader. Figure 1 below illustrates the situation and shows the construction that follows.

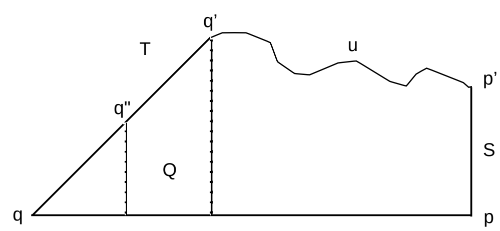

Figure 1

Let q'' be the midpoint of the segment T and consider the square Q bounded by horizontal and vertical lines drawn from q'', the horizontal axis, and a vertical line drawn from q'. Let s be the side of this square and let $s' = d(p,q) - 2s$. The last number is positive in the situation we are considering.

Suppose first that the path u enters the square Q. In this case, we project orthogonally to a vertical line and observe that the projection of u has length at least s. Also, projecting to a horizontal line, the projection of u has length at least s'. It follows that $\ell(u) \geq C_1(2s + s') \geq C_1 d(p, q)$, where $C_1 > 0$ is a constant.

Next suppose that the path u does not enter the square Q. Let P be a point in the interior of Q and consider the index of the circuit ${}_{\mathbb{Z}}$ about the point P. If the index is nonzero, then a minimal surface for ${}_{\mathbb{Z}}$ must have area at least as large as $\mathrm{Area}(Q) = s^2$. Furthermore, by projecting to the horizontal axis, we see that $\ell(u) \geq s'$. It follows that if the index is nonzero, then $d(p, q) \leq C_2(\ell(u) + A^{1/2})$ for C_2 a constant.

Finally consider the case where the index is zero, so ${}_{\mathbb{Z}}$ is null homotopic in the complement of Q. In this case it is easy to see that the projection of u on a vertical line must be at least as long as $2s$, and again the horizontal projection of u must have length at least s', so in this case we have $\ell(u) \geq C_3 d(p, q)$ for C_3 a constant.

It follows that in the our situation we have $d(p, q) \leq C(\ell(u) + A^{1/2})$ for C a constant.

One deals with the other situations analogously, and Lemma 9.4 follows.

We now consider cases (3) and (4) in Lemma 9.3. In case (4) two sides of the image of the region R_j in the Euclidean plane are bounded by vertical lines, labelled t^m and t^n, and the side corresponding to v_j is labelled y^k. The segment corresponding to u_j then has length at least $|k|$ as follows by projecting to the horizontal y-axis. No area term is needed in the conclusion of Lemma 9.3 in this case.

In case (3), two sides of the image of R_j in the plane are bounded by segments labelled $(yt)^m$ and $(yt)^n$ and one side is labelled y^k. In this case the first two sides are staircase approximations to straight lines of slope 1 at distance $|k|/\sqrt{2}$ apart. No area term is needed here either.

Thus all cases have been treated and Lemma 9.3 is established.

§10. Central extensions.

Theorem 10.1. *Let Z be an infinite cyclic central subgroup of the group E and let $\pi : E \to E/Z =: G$ be the canonical projection, where G is finitely presented. Suppose that the inclusion $Z < E$ does not distort length and that $x \in E$ is an element for which there exists a homomorphism $\phi : G \to \mathbb{Z}$ such that $\phi \circ \pi(x) \neq 0$. Then the inclusion $H =: \langle x, Z \rangle < E$ does not distort area.*

Remark. The hypothesis that $Z < E$ does not distort length is needed to exclude the example

$$1 \to Z \times 1 \to H_3 \times \langle t \rangle \xrightarrow{\pi \times 1} \mathbb{Z}^2 \times \langle t \rangle \to 1;$$

here H_3 is the 3-dimensional integral Heisenberg group, Z is its center, π is the quotient map $H_3 \to H_3/Z \cong \mathbb{Z}^2$, and $\langle t \rangle$ is an infinite cycle. The inclusion $\langle t, Z \rangle < H_3 \times \langle t \rangle$ distorts area by Theorem 8.1.

The hypothesis that the element x is detected by the homomorphism $\phi \circ \pi$ is present to exclude examples like Example 2.9 above.

Proof of Theorem 10.1. We make an initial reduction. If $\phi \circ \pi(x) = m \neq 0$, we replace E, G by their subgroups $(\phi \circ \pi)^{-1}(m), \phi^{-1}(m)$ of index m respectively. After adjusting the notation, we may assume that $\phi \circ \pi(x) = 1 \in \mathbb{Z}$, since the area distortion is not be affected by passage to a subgroup of finite index.

Let z be a generator for the infinite cyclic subgroup $Z < E$ and let $\mathcal{Q}$ be a finite presentation for E containing the subpresentation $\mathcal{P} = \langle x, z \mid [x, z]\rangle$ for $H =: \langle x, z\rangle$. We may assume in addition that all generators of $\mathcal{Q}$ except x are mapped to 0 by $\phi \circ \pi$, by making Tietze transformations on the presentation. In addition, we may assume that each relator of $\mathcal{Q}$ involving x has precisely two occurrences of x with opposite signs. This condition can be achieved by subdivision, introducing new generators each mapped to 0 by $\phi \circ \pi$ (*i.e.* we make additional Tietze transformations to achieve this last condition).

Let w be a word in the generators x, z for the subgroup $H = \langle x, z\rangle < E$ which represents the identity and let D be a minimal van Kampen diagram for w in $\mathcal{Q}$. We shall assume that D is a topological disc, since the general case can be easily reduced to this. We shall show that $\text{Area}_{\mathcal{P}}(w) \leq C\text{Area}(D)$, where $C > 0$ is a constant independent of w, from which it will follow that the inclusion $H < G$ does not distort area.

The homomorphism $\phi \circ \pi$ corresponds to a map $f : K(\mathcal{Q}) \to S^1$, where $K(\mathcal{Q})$ is the 2-complex canonically associated to the presentation $\mathcal{Q}$, and we may assume f is transverse to a general point p of S^1. The preimage $f^{-1}(p)$ is a 1-manifold I properly imbedded in D. We consider only the set of arc components $\{I_j\}$ of I in the argument that follows. Note that it is not permitted in this case to surger out the circle components of I, for that may increase the area (we do not even know that $\text{Ker}(\phi \circ \pi)$ is finitely generated, let alone finitely presented). The arc I_j determines a corridor C_j consisting of all 1 and 2 -cells of D incident with I_j. Note that C_j meets ∂D in two edges on the boundary labelled x by our assumptions of transversality and $\phi \circ \pi(x) = 1$. Furthermore, C_j does not meet any components of I other than I_j, by our assumption that each relator of $\mathcal{Q}$ has either 2 occurrences x or none. The two other sides of C_j are labelled u_j and u'_j, words in the generators of $\mathcal{Q}$. Note that both of u_j, u'_j represent the same element $z^{n_j} \in Z$. This follows since C_j separates D and makes use of the centrality of the element z.

Note that there is a positive constant M so that $\text{Area}(C_j) \geq \frac{1}{M} \min(\ell(u_j), \ell(u'_j))$ for all j. The number M can be taken to be the length of the longest relator of $\mathcal{Q}$. Note also that there is a positive constant C' so that $|n_j| \leq C'\ell(u_j)$ and $|n_j| \leq C'\ell(u'_j)$ for each index j, since the inclusion $Z < E$ does not distort lengths. Hence $|n_j| \leq C' \min(\ell(u_j), \ell(u'_j)) \leq C'M\text{Area}(C_j)$.

Next we construct a van Kampen diagram D' for w in $\mathcal{P}$ by replacing each u_j, u'_j by arcs labelled z^{n_j} and replacing C_j by a corridor C'_j with opposite sides labelled z^{n_j}. We can interpolate between these corridors with zero area, since the corresponding boundary labels are in the tree corresponding to the infinite cycle Z.

Note that the circle components of I in D play no role in this construction. It follows that $\mathrm{Area}_{\mathcal{P}}(w) \leq \mathrm{Area}(D') = \sum_j \mathrm{Area}(C'_j) = \sum_j |n_j| \leq \sum_j C'M\mathrm{Area}(C_j) \leq C'M\mathrm{Area}(D) = C\mathrm{Area}(D)$, where $C = C'M$. This completes the proof of Theorem 10.1.

Theorem 10.2. *Let the group E be a central extension of the fundamental group G of a compact hyperbolic 2-manifold by the central subgroup $Z \cong \mathbb{Z}$. Then for every abelian subgroup H of E, the inclusion $H < E$ does not distort area.*

Proof. The result is easy if G is free, since the central extension splits. So we shall assume that G is the fundamental group of a closed orientable hyperbolic 2-manifold, by passing to a subgroup of index 2 if necessary. If H is cyclic, there is nothing to prove, so we assume $H \cong \mathbb{Z}^2$. In this case, again using a finite index argument, we may assume that $H = \langle x, z\rangle$, where z generates the central subgroup Z and where $x \in E - Z$.

Let $\pi : E \to G$ be the projection, so $\pi(x) \neq 1$. We need the following

Lemma 10.3. [11] *Let G be the fundamental group of a closed surface and let $1 \neq \bar{x} \in G$. Then there is a subgroup G' of finite index in G and a homomorphism $\phi : G' \to \mathbb{Z}$ such that $\bar{x} \in G'$ and $\phi(\bar{x}) \neq 0$.*

Proof. By a theorem of G. Baumslag's, surface groups are residually free [Ba]. It follows that there is a normal subroup $N \rtimes G$ such that $\bar{x} \notin N$ and G/N is free. But M. Hall's theorem [LS] shows that the subgroup $\langle \bar{x}N\rangle$ is a free factor of a subgroup $\overline{G}$ of finite index in G/N. That is, the image of $\bar{x}N$ in $H_1(\overline{G}, \mathbb{Z}) \cong \mathbb{Z}^r$ (for some number r) is nonzero. Lifting $\overline{G}$ back to G, it follows that there is a subgroup G' of finite index of G and a homomorphism $\phi : G' \to \mathbb{Z}$ such that $\phi(\bar{x}) \neq 0$, and the proof of the Lemma is complete.

We return to the proof of the Theorem. Let G' be a subgroup of finite index of G containing $\pi(x)$ and let $\phi : G' \to \mathbb{Z}$ be a homomorphism such that $\phi(\pi(x)) \neq 0$, as guaranteed by the Lemma. Let E' be the pull-back of E via the inclusion $G' < G$. By Theorem 3.1 of [NR] E' has a biautomatic structure. It follows from a result of [GS] that the inclusion of the subgroup $Z = \langle z\rangle < E'$ does not distort length. Thus the inclusion $\langle x, z\rangle < E'$ does not distort area by Theorem 10.1. Arguing again for finite index subgroups, we deduce that the inclusion $\langle x, z\rangle < E$ does not distort area, and the proof is complete.

Theorem 10.4. *Let M be a closed geometric 3-manifold and let $H < G = \pi_1(M)$ where $H \cong \mathbb{Z}^2$. Then the inclusion $H < G$ does not distort area.*

Proof. We examine each of the eight geometries [Sc] in turn. In the cases $\rtimes^3, S^3, S^2 \times \rtimes$ there are no $\mathbb{Z}^2$ subgroups. The cases Sol and Nil follow from Theorem A above and the description of (necessarily uniform) lattices in these Lie groups [Sc]. The examples $\rtimes^2 \times \rtimes$ and $\rtimes^3$ have CAT(0) structures, so the result

[11] I am grateful to John Stallings for suggesting using Baumslag's theorem [Ba] to prove the Lemma.

follows from Example 2.10. Finally, the $\widetilde{\mathrm{Sl}}_2(\mathbb{R})$ case follows from Theorem 10.2 and the description of uniform lattices in this Lie group [Sc]. This completes the proof.

Remark. The geometries $\widetilde{\mathrm{Sl}}_2(\mathbb{R})$ and $\mathbb{H}^2 \times \mathbb{R}$ are quasi-isometric [Ge3]. However it is not immediately clear how to deduce the previous result for the former geometry from the easier result for the latter geometry, because area distortion is not *a priori* a quasi-isometry invariant property, but only an invariant of quasi-isometries adapted to pairs (*cf.* 7.2 above).

§11. Fundamental groups of 3-manifolds.

The purpose of this section is to prove the following result.

Theorem 11.1. *If A is a finitely generated abelian subgroup of $G = \pi_1(M)$, where M is a closed 3-manifold, then the inclusion $A < G$ does not distort area.*

Remark. The group A is either virtually cyclic or free abelian of rank 2 or 3 [H] Theorem 9.13.

We begin the proof of the Theorem by noting that we may assume without loss of generality that M is irreducible (*i.e.* any tame 2-sphere in M bounds a 3-ball). For A is freely indecomposable, so some conjugate must be contained in the fundamental group of one of the summands in the Kneser decomposition. We are making use here of an easy argument involving van Kampen diagrams which shows that if $A < G < G \star K$, then the inclusion $A < G$ preserves area iff the inclusion $A < G \star K$ preserves area. This also uses the fact that any homotopy 3-balls can be surgered out and replaced by honest 3-balls without changing the fundamental group.

Note also that we can assume that M is orientable by passing to a double cover if necessary. The area distortion will be unchanged in the process.

The case where A is virtually cyclic follows from Proposition A of §2 above. So we can assume that A is either a $\mathbb{Z}^2$ or $\mathbb{Z}^3$ subgroup of $G = \pi_1(M)$, where M is orientable and irreducible. The Torus Theorem ([Ga] Corollary 8.6) states that under these circumstances M either contains an imbedded π_1-injective torus or is a Seifert fibred space. The case where M is a Seifert fibred space and $A \cong \mathbb{Z}^2$ is handled by the methods of §10, in particular Theorem 10.2; if $A \cong \mathbb{Z}^3$, then A is of finite index in G and there is no area distortion.

There remains to consider the case where M contains an imbedded π_1-injective torus. In this case M is Haken and we can consider the JSJ decomposition for M [JS][Jo] (the case where M is geometric has already been treated in Theorem 10.4). If there is at least one hyperbolic geometric piece in the JSJ decomposition, then M admits a Riemannian metric of nonpositive curvature by a result of B. Leeb's thesis [L]. In this case our result follows from Example 2.10 above. Otherwise M is not geometric and does not contain any hyperbolic piece, so M is a so-called graph manifold. In this case all the pieces are Seifert fibred but M itself is not Seifert fibred. Here it is still possible for M to have a Riemannian metric of nonpositive curvature, and our result will follow from Example 2.10 as before.

It thus remains to consider the case where M is a graph manifold which does not admit any Riemannian metric of nonpositive curvature.

We call the tori occurring in the JSJ decomposition the "canonical tori" of M. They are well defined up to isotopy. We need the following result.

Proposition 11.2. *Suppose that for each canonical torus T of the graph manifold M the injective homomorphism $\pi_1(T) \to \pi_1(M) = G$ (where the base point is chosen in T) does not distort length and does not distort area. Then the inclusion $\mathbb{Z}^2 \cong A < G$ does not distort area.*

We defer the proof of Proposition 11.2 for the moment and assume this result to complete the proof of Theorem 11.1.

We shall now show that the hypotheses of Proposition 11.2 are satisfied when M is a graph manifold. By the result of [KL2], $G = \pi_1(M)$ is quasi-isometric to $G' = \pi_1(M')$, where M' is a graph manifold possessing a Riemannian metric of nonpositive curvature. Furthermore, the quasi-isometry preserves the collection of fundamental groups of canonical tori up to Hausdorff equivalence, by the result of [KL1]. But the length distortion and area distortion functions are invariants of quasi-isometries adapted to pairs, and the result of [KL1] says that this is the case here. It follows that the hypotheses of Proposition 11.2 are satisfied, and the Proposition implies Theorem 11.1.

Proof of Proposition 11.2. The JSJ decomposition of $G = \pi_1(M)$ corresponds to an action of G on a simplicial tree Γ where the edge stabilizers G_e are conjugates of the fundamental groups of the canonical tori and where the vertex stabilizers G_v are fundamental groups of the geometric pieces into which M is decomposed. We restrict the action to the subgroup $\mathbb{Z}^2 \cong A < G$ and obtain a corresponding decomposition of A. However there are only a small number of ways $\mathbb{Z}^2$ can decompose nontrivially as an amalgam or HNN extension. The possibilities are as follows.

(1) A is contained in some vertex stabilizer G_v, or
(2) $A \cap G_v$ is cyclic for all vertices v of Γ.

The first case happens iff $A \cap G_v$ is of rank 2 for some vertex v, whereas in the second case $A \cap G_v$ has rank at most 1 for all v (the second case can occur, *e.g.*, if $A = \mathbb{Z}\star_{\rtimes}$ where the base and associated subgroups are all equal).

In the first case we may assume that $A < \pi_1(P)$, where P is one of the geometric pieces. In effect this amounts to a choice of base point. Choosing appropriate presentations for A, G adapted to the action on Γ, we consider a loop w in the Cayley graph of A as imbedded in the interior of $\widetilde{P} \subset \widetilde{M}$ and we choose a minimal disc filling D of w (with respect to some Riemannian metric on M lifted equivariantly to $\widetilde{M}$). Let $\mathcal{T}$ be the union of all lifts of all canonical tori to $\widetilde{M}$, a disjoint union of copies of $\rtimes^2$. We may assume that D is transverse to $\mathcal{T}$, so the intersection of D with $\mathcal{T}$ is a 1-manifold I properly imbedded in D. Since $\mathcal{T} \cap \partial D = \emptyset$, it follows the I consists entirely of circle components in the interior. We look only at the maximal circle components of I in the nesting; these circles map to various boundary $\rtimes^2$'s of $\widetilde{P}$. So we can fill each of these maximal circles in the appropriate copy of $\rtimes^2$

and obtain thereby a new filling D' for w in $\widetilde{P}$. But by the hypothesis that there is no area distortion in mapping fundamental groups of canonical tori to G, the new filling D' has area at most a constant multiple of the area of D. On the other hand, P is Seifert fibred, so we deduce from §10 that there is no area distortion in the inclusion $A < \pi_1(P)$. Composing $A < \pi_1(P) < \pi_1(M)$, we deduce that there is no area distortion for the inclusion $A < G$.

In the second case, we imbed the Cayley graph of A in $\widetilde{M}$ and consider a loop w in it. Suppose D is a minimal area filling for w in $\widetilde{M}$. We may assume D is transverse to the collection $\mathcal{T}$ by budging it slightly. Then $D \cap \mathcal{T} = I$ is a 1-manifold properly imbedded in D. In this case we ignore the circle components of I and focus our attention on the arc components I_j of I. Now the two end points of I_j are related by translation by an element of the cyclic group $A \cap G_e$ for an appropriate edge e of Γ. So we can fill the end points in the Cayley subgraph of $A \cap G_e$, to get the new path I'_j. There is no length distortion in including cyclic subgroups of the biautomatic group $\pi_1(P)$ (where P is a geometric piece of M) and by hypothesis the metric on components of $\mathcal{T}$ is undistorted from that on $\widetilde{M}$. It follows that there is a constant $C > 0$ so that $\ell(I'_j) \leq C\ell(I_j)$ for all j.

Next we choose collar neighborhoods of the canonical tori in M of width $\epsilon > 0$ and lift them to $\widetilde{M}$. It follows that $\mathrm{Area}(D) \geq \epsilon \sum_j \ell(I_j)$, as we see by examining the induced collar neighborhoods N_j of the arcs I_j in D.

Now the boundaries of collar neighborhoods N'_j of the arcs I'_j together with portions of the boundary loop w can be used to construct a new filling D' of w in the presentation for A. Since the groups $A \cap G_v$ are cyclic, the portion of the diagram D' not involving the N'_j's is of zero area, and the area of D' is bounded by a constant times $\epsilon \sum_j \ell(I'_j) \leq C\epsilon \sum_j \ell(I_j)$. Since the last number is bounded above by a constant multiple of $\mathrm{Area}(D)$, we deduce that $\mathrm{Area}(D')$ is bounded by a constant multiple of $\mathrm{Area}(D)$. It follows that there is no area distortion for the inclusion $A < G$. This completes the proof of Proposition 11.2.

Corollary 11.3. *If A is a finitely generated abelian subgroup of $G = \pi_1(M)$, where M is a compact 3-manifold, then the inclusion $A < G$ does not distort area.*

Proof. If M is bounded, we double M along its boundary to form the closed 3-manifold N. The homomorphism $\pi_1(M) \to \pi_1(N)$ induced by inclusion is injective, since N retracts onto M. By Theorem 11.1, the composition $A < G \rightarrowtail \pi_1(N)$ does not distort area, from which it follows that $A < G$ does not distort area.

Remark. In the proof of Theorem 11.1 we made use of Thurston's geometrization *theorem*, which states that the pieces of the JSJ decomposition of a compact Haken 3-manifold are geometric, but we did not assume his *conjecture* (which states that those pieces of a general compact 3-manifold are geometric) at any point in the argument. Theorem 11.1 gives no information about a torsion-free rank 1 abelian subgroup A of a compact 3-manifold group G nor about the pair (G, A). A consequence of Thurston's geometrization conjecture is that such a subgroup A is finitely generated.

§12. Open questions.

12.1. Since the 3-dimensional integral Heisenberg group H_3 has a cubic Dehn function [ECHLPT][Ge2], the exponent $\alpha(G, H_3)$ of area distortion for an imbedding $H_3 < G$ in a finitely presented group is a real number between 1 and 3. In particular one can consider the standard imbedding $H_3 < H_5$ of H_3 in the 5-dimensional Heisenberg group. It follows from Example 2.8 above that $\alpha(H_5, H_3) \geq 3/2$. It is natural to ask whether in fact one has $\alpha(H_5, H_3) = 3/2$.

12.2. A basic question is to ask for the classification of quasi-isometry types of groups $\mathbb{Z}^n \rtimes_\phi \mathbb{Z}$. The integer n is a quasi-isometry invariant [BG], so one may ask, for example, whether or not two such groups with the same n and PV monodromies are quasi-isometric. One asks also how many quasi-isometry classes of pairs $(\mathbb{Z}^n \rtimes_\phi \mathbb{Z}, \mathbb{Z}^n)$ there are for fixed n and PV monodromy. Similarly one can raise the same questions when the monodromy ϕ is hyperbolic, in the sense of having no eigenvalues on the unit circle.

12.3. Suppose that M is a compact 3-manifold and H is a finitely generated subgroup of $G = \pi_1(M)$. Is it possible for the inclusion $H < G$ to distort area?

12.4. If G is a biautomatic group and $H < G$ where $H \cong \mathbb{Z}^2$, can the inclusion $H < G$ distort area? Note that H is quasi-isometrically imbedded in G, as follows from results of [GS]. However I gave an example in §4 above of a quasi-isometrically imbedded subgroup of a biautomatic group where the inclusion homomorphism distorts area (in the notation of §4 the nonautomatic subgroup H of the group G was quasi-isometrically imbedded although the area was distorted); so I am asking now whether this pathology can occur when $H \cong \mathbb{Z}^2$.

12.5. Suppose that $\mathcal{G}$ is a 1-connected solvable Lie group with nilradical $\mathcal{N}$. If G is a lattice in $\mathcal{G}$, then $G \cap \mathcal{N}$ is a lattice in $\mathcal{N}$ and the exponent of area distortion $\alpha = \alpha(G, G \cap \mathcal{N})$ is defined, since the nilpotent group $G \cap \mathcal{N}$ satisfies a polynomial isoperimetric inequality. It follows from Proposition 7.7 that α takes the same value at all lattices of $\mathcal{G}$. It is of interest to develop methods to calculate α. Also I want to know whether the pair $(G, G \cap \mathcal{N})$ is *rigid*, *i.e.* whether every quasi-isometry of G with itself is adapted to the pair $(G, G \cap \mathcal{N})$ in the sense of §7.

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