CHAPTER 16: Confidence Intervals: The Basics

Basic Practice of Statistics

8th Edition

Lecture PowerPoint Slides

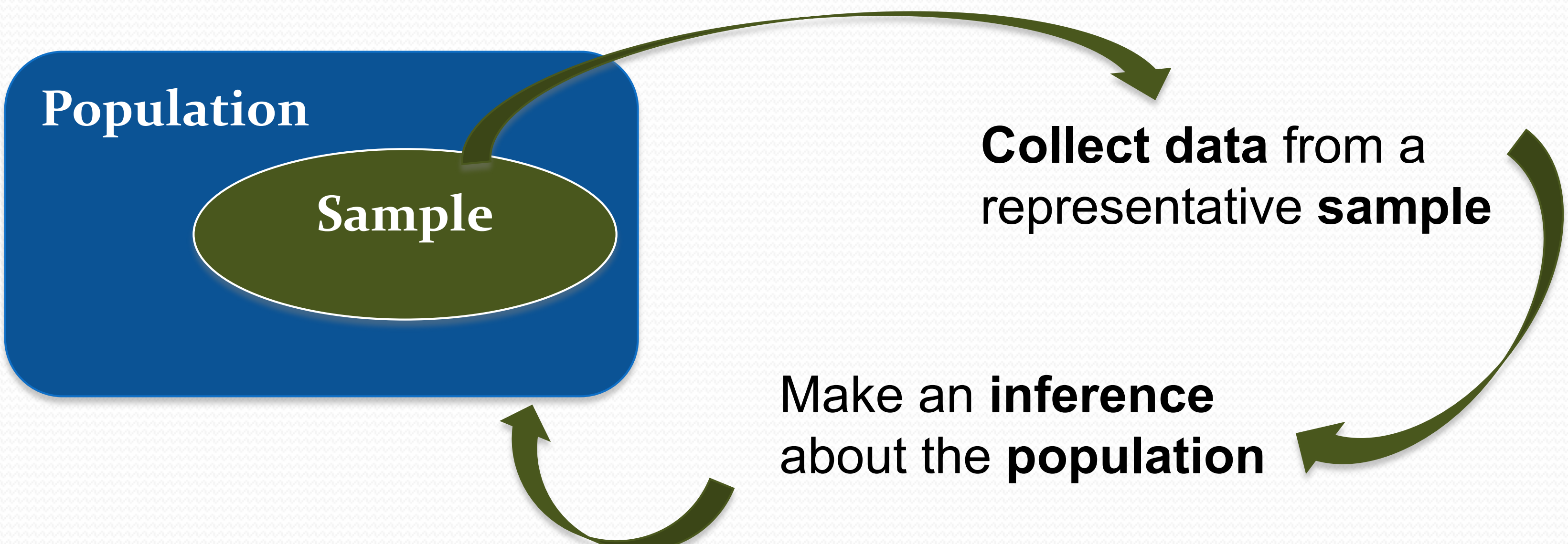
In Chapter 16, we cover …

- The reasoning of statistical estimation
- Margin of error and confidence level
- Confidence intervals for a population mean
- How confidence intervals behave

Statistical inference

- After we have selected a sample, we know the responses of the individuals in the sample. However, the reason for taking the sample is to infer from that data some conclusion about the wider population represented by the sample.

- Statistical inference provides methods for drawing conclusions about a population from sample data.



Simple conditions for inference about a mean

- This chapter presents the basic reasoning of statistical inference. We start with a setting that is too simple to be realistic.

SIMPLE CONDITIONS FOR INFERENCE ABOUT A MEAN

1. We have a simple random sample (SRS) from the population of interest. There is no nonresponse or other practical difficulty. The population is large compared to the size of the sample.
2. The variable we measure has an exactly Normal distribution $N(\mu,\ \sigma)$ in the population.
3. We don't know the population mean μ, but we do know the population standard deviation σ.

Note: The conditions that we have a perfect SRS, that the population is exactly Normal.

Statistical Inference (1 of 4)

What is meant by statistical inference?

a) drawing conclusions about a population based on information in a sample
b) drawing conclusions about a sample based on information contained in a population
c) drawing conclusions about a sample based on the measurements in that sample
d) selecting a set of data from a large population

Statistical Inference (2 of 4)

- The simple conditions for introducing basic ideas about inference for μ do not include:

a) a simple random sample.

b) a measure of interest that follows a Normal distribution exactly.

c) a known value of the population mean μ.

d) a known value of the population standard deviation σ.

Statistical Inference (3 of 4)

- Why might we need the population to follow the Normal distribution in order to do inference for μ?

a) so that the distribution of the sample data is Normal

b) so that $\bar{x}$ equals μ

c) so that σ is known

d) so that the sampling distribution of $\bar{x}$ is Normal

Statistical Inference (4 of 4)

- When doing inference for μ, it is common that σ is known.

a) true

b) false

Point Estimator

By the Law of Large Numbers:

- The sample mean is a good estimate of the true mean.

The sample mean is a "point estimator."
It estimates the value of the parameter.

But how "confident" are we?

The reasoning of statistical estimation (part I)

An NHANES report gives data for 654 women aged 20 to 29 years. The mean BMI of these 654 women is $\bar{x} = 26.8$. On the basis of this sample, we want to estimate the mean BMI μ in the population of all 20.6 million women in this age group. To match the "simple conditions," we will treat the NHANES sample as an SRS from a Normal population with known standard deviation $\sigma = 7.5$.

1. To estimate the unknown population mean BMI μ, use the mean $\bar{x} = 26.8$ of the random sample. We don't expect $\bar{x}$ to be exactly equal to μ, so we want to say how accurate this estimate is.

The reasoning of statistical estimation (part II)

2. The average BMI $\bar{x}$ of an SRS of 654 young women has standard deviation $\sigma/\sqrt{n} = 7.5/\sqrt{654} = 0.3$, rounded.
3. The "95" part of the 68–95–99.7 rule for Normal distributions says that $\bar{x}$ is within 0.6 (2 standard deviations) of its mean, μ, in 95% of all samples. So if we construct the interval $[\bar{x} - 0.6,\ \bar{x} + 0.6]$ and estimate that μ lies in the interval, we will be correct 95% of the time.
4. Adding and subtracting 0.6 from our sample mean of 26.8, we get the interval [26.2, 27.4]. For this we say that we are 95% confident that the mean BMI, μ, of all young women is some value in that interval—no lower than 26.2 and no higher than 27.4.

Reasoning of Estimation (1 of 4)

- Why is it sensible to use $\bar{x}$ as a guess for μ?

a) $\bar{x}$ is always equal to μ.

b) $\bar{x}$ is the same from sample to sample.

c) The sampling distribution of $\bar{x}$ tells us how close $\bar{x}$ is likely to be to μ.

d) $\bar{x}$ is the easiest statistic to calculate.

Reasoning of Estimation (2 of 4)

- In repeated sampling, $\bar{x}$ should be within ______ of μ 68% of the time.

a) 2σ

b) $\pm\, 2\sigma / \sqrt{n}$

c) σ

d) $\pm\, \sigma / \sqrt{n}$

Reasoning of Estimation (3 of 4)

- The purpose of a confidence interval for μ is to give a range of likely values for:

a) the level of confidence.

b) the sample mean.

c) the unknown population mean.

d) the difference between the sample mean and the population mean.

Reasoning of Estimation (4 of 4)

What do we hope is within a confidence interval?

a) the unknown confidence level
b) the unknown parameter
c) the unknown statistic
d) the parameter estimate
e) the margin of error
f) the sample size

Confidence interval

- In our previous example, the 95% confidence interval was $\bar{x} \pm 0.6$.
- Most confidence intervals will have a form similar to this:

$$\text{estimate} \pm \text{margin of error}$$

- The **margin of error** $\pm\, 0.6$ shows how accurate we believe our guess is; the margin being based on the variability of the estimate.

- A level *C* confidence interval for a parameter has two parts:
  - An interval calculated from the data, which has the form:

$$\text{estimate} \pm \text{margin of error}$$

  - A confidence level *C*, which gives the probability that the interval will capture the true parameter value in repeated samples. That is, the confidence level is the success rate for the method.

Confidence level

- The confidence level is the overall capture rate if the method is used many times. The sample mean will vary from sample to sample, but when we use the method $\text{estimate} \pm \text{margin of error}$ to get an interval based on each sample, *C*% of these intervals capture the unknown population mean μ.

INTERPRETING A CONFIDENCE LEVEL

- The confidence level is the success rate of the method that produces the interval. We don't know whether the 95% confidence interval from a particular sample is one of the 95% that capture μ or one of the unlucky 5% that miss.
- To say that we are **95% confident** that the unknown μ lies between 26.2 and 27.4 is shorthand for “We got these numbers using a method that gives correct results 95% of the time.”

Confidence level illustrated (part I)

SRS size $n = 654$ $\longrightarrow$ $\bar{x} \pm 0.6 = 26.8 \pm 0.6$

SRS size $n = 654$ $\longrightarrow$ $\bar{x} \pm 0.6 = 27.0 \pm 0.6$

SRS size $\mathrm{n} = 654$ $\longrightarrow$ $\bar{x} \pm 0.6 = 27.2 \pm 0.6$

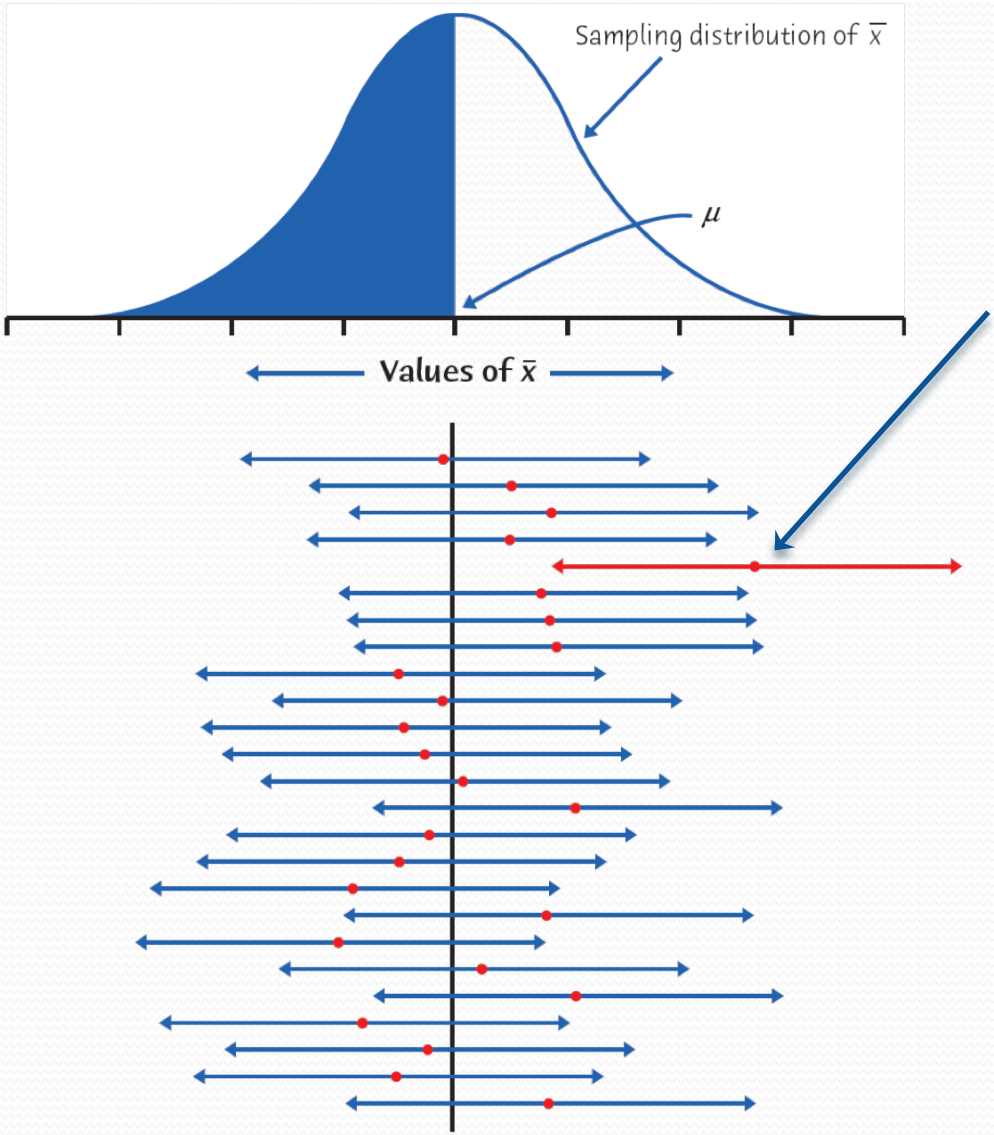
$\vdots$ $\vdots$

MANY SRSs MANY CONFIDENCE INTERVALS

POPULATION,
mean μ unknown,
$\sigma = 75$

95% of these intervals capture the unknown mean μ of the population

Confidence level illustrated (part II)



This interval misses the true mean μ. The others all capture μ.

Case Study

NAEP Quantitative Scores
(National Assessment of Educational Progress)

Rivera-Batiz, F. L., “Quantitative literacy and the likelihood of employment among young adults,” *Journal of Human Resources*, **27** (1992), pp. 313-328.

What is the average score for all young adult males?

Case Study

NAEP Quantitative Scores

The NAEP survey includes a short test of quantitative skills, covering mainly basic arithmetic and the ability to apply it to realistic problems. Scores on the test range from 0 to 500, with higher scores indicating greater numerical abilities. It is known that NAEP scores have standard deviation $\sigma = 60$.

Case Study

NAEP Quantitative Scores

In a recent year, 840 men 21 to 25 years of age were in the NAEP sample. Their mean quantitative score was 272.

On the basis of this sample, estimate the mean score μ in the population of all 9.5 million young men of these ages.

Case Study

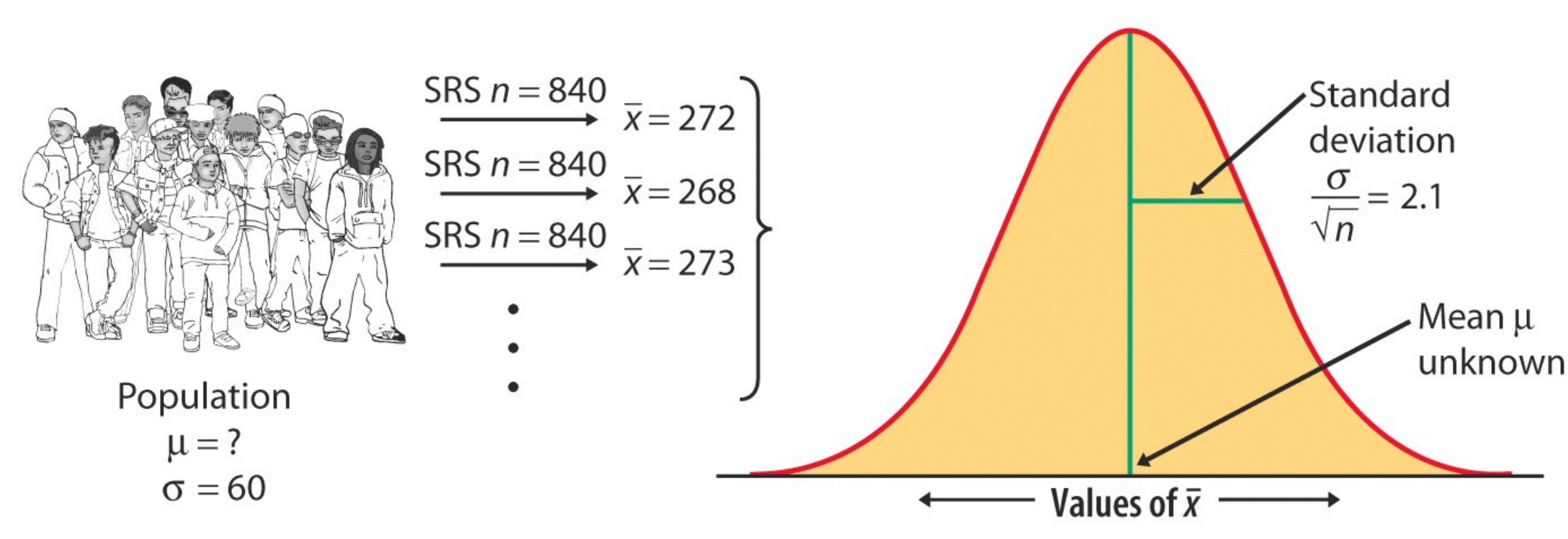
NAEP Quantitative Scores

1. To estimate the unknown population mean μ, use the sample mean $\bar{x}$ = 272.
2. The law of large numbers suggests that $\bar{x}$ will be close to μ, but there will be some error in the estimate.
3. The sampling distribution of $\bar{x}$ has the Normal distribution with mean μ and standard deviation

$$\frac{\sigma}{\sqrt{n}} = \frac{60}{\sqrt{840}} \approx 2.1$$

Case Study

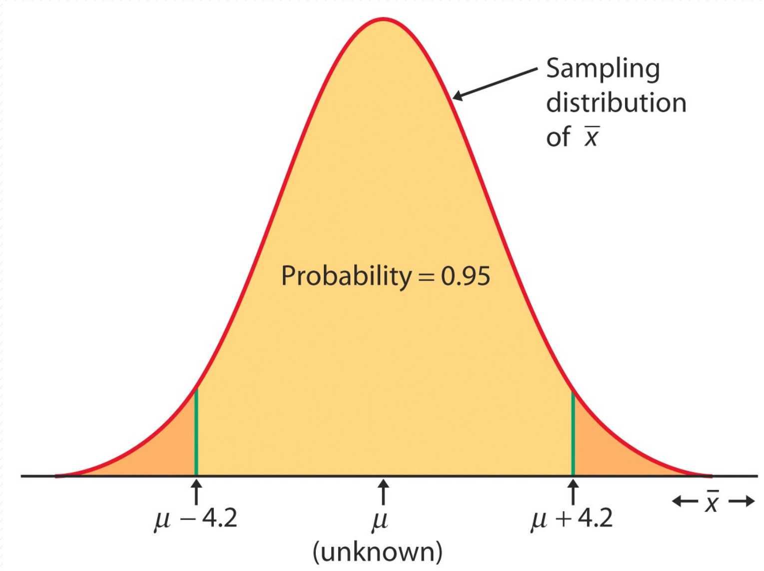
NAEP Quantitative Scores



Case Study

NAEP Quantitative Scores

4. The 68-95-99.7 rule indicates that $\bar{x}$ and μ are within two standard deviations (4.2) of each other in about 95% of all samples.

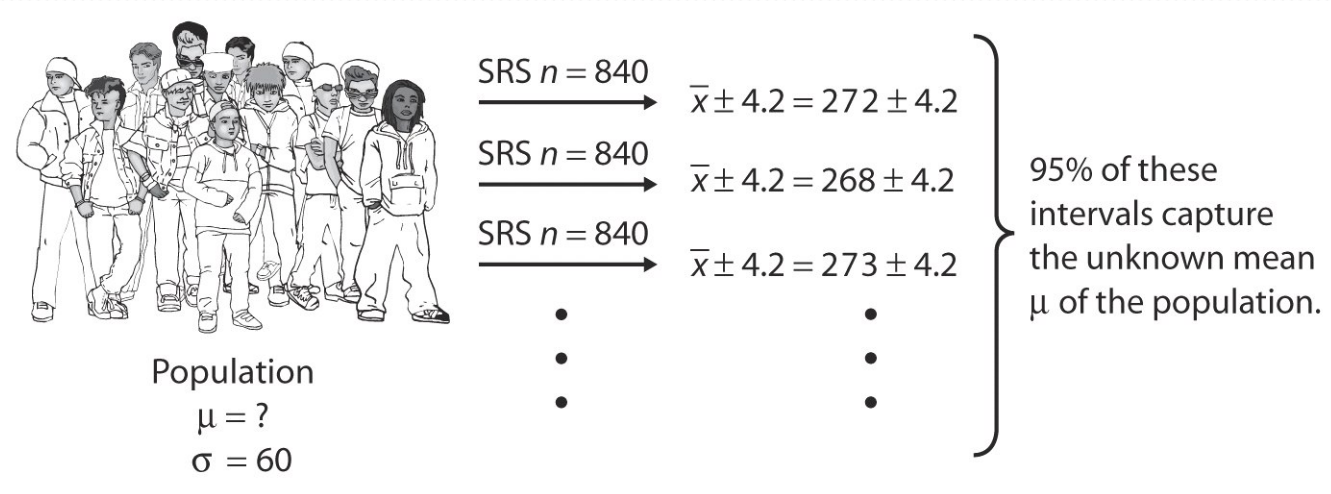


$$\bar{x} - 4.2 = 272 - 4.2 = 267.8$$
$$\bar{x} + 4.2 = 272 + 4.2 = 276.2$$

Case Study

NAEP Quantitative Scores

So, if we estimate that μ lies within 4.2 of $\bar{x}$, we'll be right about 95% of the time.



Careful Interpretation of a Confidence Interval

- ***"We are 95% confident that the mean NAEP score for the population of all adult males is between 267.884 and 276.116."***

  (We feel that plausible values for the population of males' mean NAEP score are between 267.884 and 276.116.)
- ** This *does not mean* that 95% of all males will have NAEP scores between 267.884 and 276.116. **
- *Statistically*: 95% of the C.I.s should contain the true population mean.
- Again: 5% of the polls you've heard so far were wrong!

Margin of Error and Confidence Level (1 of 3)

What are three components of a confidence interval?

a) estimate of confidence level, sample size, and margin of error
b) mean of sample statistic, confidence level, and margin of error
c) estimate of population parameter, confidence level, and margin of error

Margin of Error and Confidence Level (2 of 3)

- The statistic used to estimate a parameter was –0.6. The confidence interval for the parameter went from –0.8 to –0.4. What was the ME for the statistic?

a) 0.1

b) 0.2

c) 0.3

d) 0.4

Margin of Error and Confidence Level (3 of 3)

The confidence level is:

a) the same as the margin of error.

b) always 95%.

c) the success rate for the method.

d) always 90%.

Confidence interval for	conditions for use	Confidence interval formula: **estimate$\pm$margin of error**
the population mean μ (When σ is known)	an SRS of size n from a Normal population	$\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}}$
the population mean μ (When σ is unknown)	an SRS of size n from a Normal population or a large population	$\bar{x} \pm t^* \frac{s}{\sqrt{n}}$
comparing two population means $\mu_1 - \mu_2$	an SRS of size n_1 from a large Normal population with μ_1 and an independent SRS of size n_2 from another large Normal population with μ_2	$(\bar{x}_1 - \bar{x}_2) \pm t^* \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$
a population proportion p	an SRS of size n from a large population	$\hat{p} \pm z^* \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
comparing two population proportions $p_1 - p_2$	an SRS of size n_1 from a large population with proportion p_1 of success and an independent SRS of size n_2 from another large population with proportion p_2 of success	$(\hat{p}_1 - \hat{p}_2) \pm z^* \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$

Confidence intervals for a population mean

- In our NHANES example, wanting "95% confidence" dictated going out 2 standard deviations in both directions from the mean—if we change our confidence level *C*, we will change the number of standard deviations. The text includes a table with the most common multiples:

Confidence level C	90%	95%	99%
Critical value z*	1.645	1.960	2.576

- Once we have these, we may build any level *C* confidence interval we wish.

CONFIDENCE INTERVAL FOR THE MEAN OF A NORMAL POPULATION

- Draw an SRS of size n from a Normal population having unknown mean μ and known standard deviation σ. A level *C* confidence interval for μ is:

$$\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}}$$

- Some examples of critical values, z^*, corresponding to the confidence level *C* are given above.

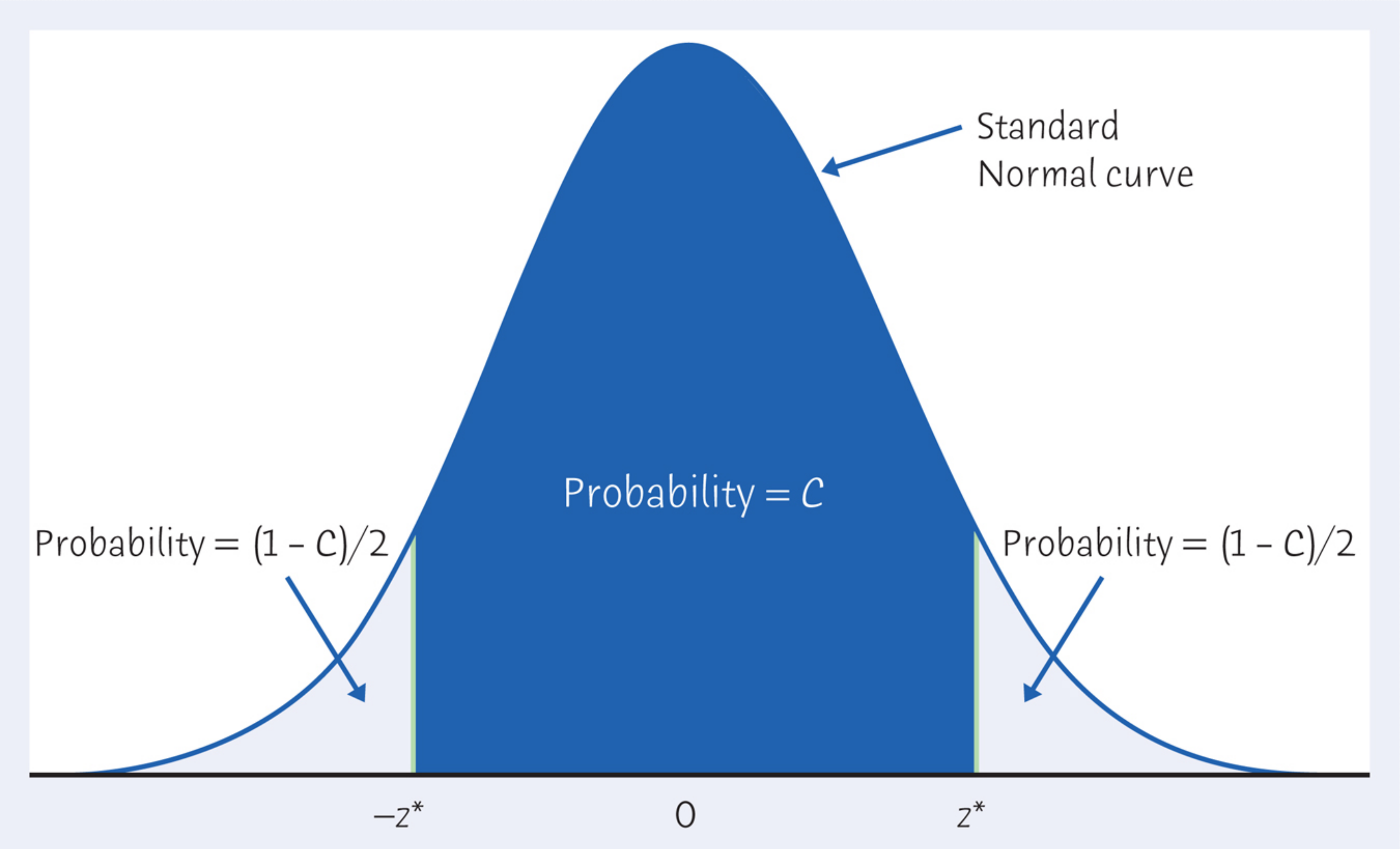


Figure 16.3, The Basic Practice of Statistics, © 2015 W. H. Freeman

$z*$: Finding critical value by using the bottom of Table C

Confidence level C	90%	95%	99%
Critical value z^*	1.645	1.960	2.576

Confidence intervals: the four-step process

- The steps in finding a confidence interval mirror the overall four-step process for organizing statistical problems.

THE FOUR-STEP PROCESS

- **State**: What is the practical question that requires estimating a parameter?
- **Plan**: Identify the parameter, choose a level of confidence, and select the type of confidence interval that fits your situation.
- **Solve**: Carry out the work in two phases:

  1. Check the conditions for the interval that you plan to use.

  2. Calculate the confidence interval.
- **Conclude**: Return to the practical question to describe your results in this setting.

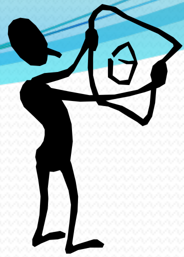
How confidence intervals behave

- The z confidence interval for the mean of a Normal population illustrates several important properties that are shared by all confidence intervals in common use: the user chooses the confidence level and the margin of error follows; we would like high confidence and a small margin of error; high confidence suggests our method almost always gives correct answers; and a small margin of error suggests we have pinned down the parameter precisely.

How do we get a small margin of error?

- The margin of error for the *z* confidence interval is: $z^* \frac{\sigma}{\sqrt{n}}$.
- The margin of error gets smaller when:
  - z^* gets smaller (the same as a lower confidence level C).
  - σ is smaller—it is easier to pin down μ when σ is smaller.
  - n gets larger—since n is under the square root sign, we must take four times as many observations to cut the margin of error in half.

Case Study



NAEP Quantitative Scores (Ch. 14)

95% Confidence Interval

$$\bar{x} - (1.960)(2.1) = 272 - 4.116 = 267.884$$

$$\bar{x} + (1.960)(2.1) = 272 + 4.116 = 276.116$$

90% Confidence Interval

$$\bar{x} - (1.645)(2.1) = 272 - 3.4545 = 268.5455$$

$$\bar{x} + (1.645)(2.1) = 272 + 3.4545 = 275.4545$$

The 90% CI is narrower than the 95% CI.

Planning Studies

Choosing the Sample Size for a C.I.

The confidence interval for the mean of a Normal population will have a specified margin of error *m* when the sample size is:

$$n = \left(\frac{z^* \sigma}{m} \right)^2$$

Case Study

NAEP Quantitative Scores (Ch.14)

Suppose that we want to estimate the population mean NAEP scores using a 90% confidence interval, and we are instructed to do so such that the margin of error does not exceed 3 points (recall that $\sigma = 60$).

What sample size will be required to enable us to create such an interval?

Case Study

NAEP Quantitative Scores

$$n = \left(\frac{z^{*}\sigma}{m}\right)^2 = \left(\frac{(1.645)(60)}{3}\right)^2 = 1082.41$$

Thus, we will need to sample at least 1082.41 men aged 21 to 25 years to ensure a margin of error not to exceed 3 points.

Note that since we can't sample a fraction of an individual and using 1082 men will yield a margin of error slightly more than 3 points, our sample size should be $n = 1083$ men.

Cautions About Confidence Intervals

The margin of error does not cover all errors.

- The margin of error in a confidence interval covers only random sampling errors. No other source of variation or bias in the sample data influence the sampling distribution.
- Practical difficulties such as undercoverage and nonresponse are often more serious than random sampling error. The margin of error does not take such difficulties into account.

Be aware of these points when reading any study results.

Confidence Interval for μ (1 of 10)

- A large school district in Connecticut wants to estimate the average SAT score of this year's graduating class using a 95% confidence interval. The district takes a simple random sample of 100 seniors. The sample mean is 510 and the population standard deviation for SAT scores is known to be 75. What is the 95% confidence interval for μ?

a) $510 \pm 1.96(75)$
b) $510 \pm (75/10)$
c) $510 \pm 1.96(75)(10)$
d) $510 \pm 1.96(75/10)$

Confidence Interval for μ (2 of 10)

A large school district in Connecticut wants to estimate the average SAT score of this year's graduating class using a confidence interval. The district takes a simple random sample of 100 seniors. The sample mean is 510 and the population standard deviation for SAT scores is known to be 75. They want an interval that "likely" contains the true mean, where "likely" means that the procedure has a 68% success rate. What is the margin of error for this confidence interval?

a) 150
b) 15
c) 7.5
d) 0.75

Confidence Interval for μ (3 of 10)

- What is the confidence level in a confidence interval for μ?

a) the percent of confidence intervals produced by the procedure that contain μ

b) the probability that a specific confidence interval contains μ

c) the percent of confidence interval procedures that will create an interval that contains μ

Confidence Interval for μ (4 of 10)

- The probability that a specific 90% confidence interval for μ actually contains μ is:

a) 0.

b) 0.1.

c) 0.9.

d) not defined.

Confidence Interval for μ (5 of 10)

Evaluate the following statement about a confidence interval: "The average time a local company takes to process new insurance claims is 9 to 11 days."

a) The statement is incomplete, because there is no description of the target parameter.
b) The statement is incomplete, because the interval is not reported.
c) The statement is incomplete, because the confidence level is not reported.
d) The statement is a complete description of the confidence interval.

Confidence Interval for μ (6 of 10)

A very large school district in Connecticut wants to estimate the average SAT score of this year's graduating class. The district takes a simple random sample of 100 seniors and calculates the 95% confidence interval for the graduating students' average SAT score at 505 to 520 points. They report that, for the sample of 100 graduating seniors, 95% of the SAT scores were between 505 and 520 points.

a) This is a correct interpretation of the interval.

b) This is an incorrect interpretation of the interval.

Confidence Interval for μ (7 of 10)

We are 90% confident that the interval from 119.5 to 128.1 bushels per acre contains the true mean yield.

a) This is a correct interpretation of the interval.
b) This is an incorrect interpretation of the interval.

Confidence Interval for μ (8 of 10)

Ninety-nine percent of the time, mean IQs of seventh-grade girls will belong to the interval (95.3, 109.2).

a) This is a correct interpretation of the interval.
b) This is an incorrect interpretation of the interval.

Confidence Interval for μ (9 of 10)

The mean IQ of all seventh-grade girls in the school district is between 95.3 and 109.2, with 99% confidence.

a) This is a correct interpretation of the interval.
b) This is an incorrect interpretation of the interval.

Confidence Interval for μ (10 of 10)

Ninety-nine percent of the IQs are between 95.3 and 109.2.

a) This is a correct interpretation of the interval.
b) This is an incorrect interpretation of the interval.

Margin of error

The margin of error depends on the sample mean, $\bar{x}$.

a) true

b) false

How Confidence Intervals Behave (1 of 4)

Increasing the confidence level will:

a) increase the margin of error.

b) decrease the margin of error.

How Confidence Intervals Behave (2 of 4)

Increasing the sample size will:

a) increase the margin of error.
b) decrease the margin of error.

How Confidence Intervals Behave (3 of 4)

Increasing the standard deviation will:

a) increase the margin of error.

b) decrease the margin of error.

How Confidence Intervals Behave (4 of 4)

It is possible to calculate a sample size (n) for a desired (pre-defined) margin of error.

a) true

b) false