WS #6 – MATH 6320
SPRING 2015

DUE: THURSDAY APRIL 9TH

You are encouraged to work in pairs on this.
First let's do some computations by hand.
1. Compute a free resolution of the ideal $\langle x, y\rangle$ inside $R = k[x, y]$.

2. Use this to try to compute directly by hand $\mathrm{Tor}_i(\langle x, y\rangle, k[x, y]/\langle x, y\rangle)$.

3. Likewise compute directly $\mathrm{Ext}^i(\langle x, y\rangle, k[x, y]/\langle x, y\rangle)$.

4. Suppose $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ and $0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$ are short exact sequences of R-modules and that $K \subseteq N_2$ is a submodule. Suppose that $\mathrm{Ext}^3(M_1, N_1) \longrightarrow \mathrm{Ext}^3(M_1, N_2/K)$ is injective, $\mathrm{Ext}^3(M_3, N_1) \longrightarrow \mathrm{Ext}^3(M_2, N_1)$ is also injective and $\mathrm{Ext}^2(M_1, N_1) = \mathrm{Ext}^2(M_3, N_1) = 0$.

Show that $\mathrm{Ext}^2(M_1, N_2) \longrightarrow \mathrm{Ext}^2(M_1, N_3)$ is an isomorphism.

Start Macaulay2 by running `M2` in a terminal. Then run `setup` (answer yes) and finally `exit` Macaulay2 and run `emacs`. You should be able to start Macaulay2 by hitting f12 within emacs.

Once you have Macaulay2 running, we begin by defining rings:

```
R = CC[x,y]
S = QQ[a,b]
T = ZZ/5[u,v]
```

For example. Rings over finite fields tend to have faster computations (for obvious reasons). It's probably better to work with one ring at a time (unless you need more). You can always run `restart` to reset everything.

We can then define ideals:

```
I = ideal(u^2-v^3, u^3+u*v^2+v^7)
J = ideal(u^3, u^2*v, u*v^2, v^3)
```

We can add, multiply or intersect them:

```
I + J
I*J
intersect(I, J)
```

We can compare whether they are contained in each other

```
isSubset(I*J, intersect(I, J))
isSubset(intersect(I,J), I*J)
```

5. Do the answers you got coincide with what you expect?

Now let's explore modules.

Begin by running `restart` and then `R = QQ[x,y,z]`.

While R is a free rank-1 R-module, Macaulay2 does not treat it as such. To consider a free R-module of rank 5 you can create:

```
M = R^5
```

Of course, many modules are also presented with relations, these are cokernels of module maps. So we begin by making a matrix, a map between free modules.

```
m = matrix{ {x, x*y, z^2}, {z, z*x, y} }
L = coker m
N = ker m
```

6. Create a matrix giving a presentation of the ideal $\langle x, y\rangle \subset \mathbb{Q}[x, y]$ in Macaulay2. Likewise create a matrix giving a presentation of $\mathbb{Q}[x, y]/\langle x, y\rangle$. Call the associated modules/cokernels `M` and `N` respectively in Macaulay2.

7. Compute

```
Ext^i(M, N), Tor_i(M, N)
```

Do these agree with your computations on the first page? (Take some time to try to make sense of the output Macaulay2 gives you, do the modules it produces really coincide with your answers?)

We earlier had the problem of finding the relations between the generators of $k[x(x-1), x^2(x-1)]$. Macaulay2 can help us with this kind of issue. First write down a guess as to what the relation between $a = x(x-1)$ and $b = x^2(x-1)$ is.

8. `restart` Macaulay2 first, then construct `R = QQ[x], S = QQ[a,b]`. Use the command `f = map(R, S, { ..., ... } )` to construct a ring map between R and S. The entries in the dots (...) should be where `a` and `b` are sent (try `viewHelp map` if you are confused). Compute `ker f` in Macaulay2, does this coincide with your guess?

9. `restart` Macaulay2 again. Let $R = \mathbb{Q}[x, y]$ and map between two free modules $\phi : R^3 \longrightarrow R^2$ such that the cokernel is not a free module (and is nonzero). You can use Macaulay2 to help (`prune` should try to simplify your module, if it is *graded*, it should tell you that the module is free). You can try (in vain) to find an example where the kernel is not free, but you'd need more variables for that. For a challenge, find an example the kernel is non-free for a polynomial ring with more variables.

10. Let $\phi : R^3 \longrightarrow R^2$ be the map you constructed in 9. Fix $M = \langle x^2 + y^3 \rangle$. Find presentations of the kernel and cokernel of the induced maps

$$\mathrm{Hom}_R(M, R^3) \longrightarrow \mathrm{Hom}_R(M, R^2)$$

and

$$M \otimes R^3 \longrightarrow M \otimes R^2.$$

Hint: You'll have to figure out how to apply Hom and $\otimes$ to maps of modules in Macaulay2, it isn't as hard as you might think.