WS #5 – MATH 6320
SPRING 2015

DUE: TUESDAY MARCH 31ST

You are encouraged to work in groups on this. Only one write up needs to be turned in per group.

1. Given modules $B, \ldots, D, B', \ldots, D'$ and R-module maps $b, \ldots, d, a', \ldots, c', \beta, \ldots, \gamma$ as in the commutative diagram below:

$$\begin{tikzcd} & B \arrow[r, "b"] \arrow[d, "\beta"] & C \arrow[r, "c"] \arrow[d, "\gamma"] & D \arrow[r, "d"] \arrow[d, "\delta"] & 0 \\ 0 \arrow[r, "a'"'] & B' \arrow[r, "b'"'] & C' \arrow[r, "c'"'] & D' & \end{tikzcd}$$

If the rows are exact, then we have an exact sequence

$$\ker\beta \longrightarrow \ker\gamma \longrightarrow \ker\delta \xrightarrow{\phi} \mathrm{coker}\beta \longrightarrow \mathrm{coker}\gamma \longrightarrow \mathrm{coker}\delta$$

where the maps between the kernels and cokernels are induced by b, c, b', c' and the map ϕ is black magic (in the proof, the exactness that I really want to read about is the exactness at ϕ).

2. Suppose that $P_\bullet, P'_\bullet$ are projective resolutions of R-modules M and N. Suppose have an R-module map $h : M \longrightarrow N$. Consider the diagram

$$\begin{tikzcd}
\cdots \arrow[d] & \cdots \arrow[d] \\
P_2 \arrow[r, dotted] \arrow[d, "d"'] & P'_2 \arrow[d, "d'"] \\
P_1 \arrow[r, dotted] \arrow[d] & P'_1 \arrow[d] \\
P_0 \arrow[r, dotted] \arrow[d, two heads] & P'_0 \arrow[d, two heads] \\
M \arrow[r, "h"'] & N
\end{tikzcd}$$

Show that there exist the dotted arrows making the diagram commute. This gives us a map of complexes $\phi : P_\bullet \longrightarrow P'_\bullet$. Was the map you constructed unique?

Definition: Two maps of complexes $\phi, \psi : K_\bullet \longrightarrow L_\bullet$ are declared to be *homotopic* (denoted $\phi \sim \psi$ frequently) if for each n, there is an R-module map $h_n : K_n \longrightarrow L_{n+1}$ such that

$$\phi_n - \psi_n = d_L h_n + h_{n-1} d_K$$

3. If ϕ, ψ are homotopic, show that they induce the same maps on homology.

4. Back in the setting of **2.**, show that any two maps $\phi, \psi : P_\bullet \longrightarrow P'_\bullet$ you constructed are homotopic.

5. Conclude that this homotopy is preserved after tensoring by another module A and thus conclude that $\mathrm{Tor}_i(M, A)$ is independent of the choice of projective resolution of M.