HW #6 – MATH 6320
SPRING 2015

DUE: THURSDAY APRIL 2ND

(1) Suppose that R is a Noetherian ring. Is it true that there exists an integer $n_0 > 0$ such that
   (a) every ideal $I \subseteq R$ is generated by at most n_0 elements?
   (b) every ascending chain of ideals $I_1 \subsetneq I_2 \subsetneq \dots$ has length at most n_0?

   Prove or give a counter example.

(2) Let $A = k[x, y]$, $I = \langle x \rangle$ and $B = k$. We have the natural projection $f : A \longrightarrow A/I$ and the natural inclusion $g : k \hookrightarrow A/I \cong k[y]$. Consider the ring $C = \{(a, b) \in A \oplus B \mid f(a) = g(b)\}$ as in problem (8) from homework 5.
   (a) Show that projection the map from $i : C \longrightarrow A$ is injective
   (b) Show that C is non-Noetherian even though it is a subring of a Noetherian ring.
   (c) Describe the maximal ideal of C in terms of the map $i^\sharp : \mathrm{Spec} A \longrightarrow \mathrm{Spec} C$.

   Hint: You may assume (8) from the previous homework.

(3) Show that every finitely generated module over a Noetherian ring is finitely presented.

(4) Show that every Artinian integral domain is a field.

(5) An R-module I is called *injective* if for every *injective* map of R-modules $f : A \longrightarrow B$ and every R-module map $g : A \longrightarrow I$, there exists a map $h : B \longrightarrow I$ such that the following diagram commutes:

$$\begin{tikzcd} I & \\ A \arrow[u, "g"] \arrow[r, "f"'] & B. \arrow[ul, dotted, "h"'] \end{tikzcd}$$

   (a) Show that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module, as is $\mathbb{Q}/\mathbb{Z}$.
   (b) Suppose that $0 \longrightarrow I \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact, prove that

$$0 \longrightarrow \mathrm{Hom}_R(N, I) \longrightarrow \mathrm{Hom}_R(N, B) \longrightarrow \mathrm{Hom}_R(N, C) \longrightarrow 0$$

   is exact for every R-module N if I is injective.

(6) Prove the 5-lemma. In other words, suppose that

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\alpha_1"] & M_2 \arrow[r] \arrow[d, "\alpha_2"] & M_3 \arrow[r] \arrow[d, "\alpha_3"] & M_4 \arrow[r] \arrow[d, "\alpha_4"] & M_5 \arrow[r] \arrow[d, "\alpha_5"] & 0 \\ 0 \arrow[r] & N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & N_4 \arrow[r] & N_5 \arrow[r] & 0 \end{tikzcd}$$

   is a commutative diagram with exact sequences as rows.
   (a) If we suppose that α_2 and α_4 are surjective, and α_5 is injective, show that α_3 is surjective.
   (b) If we suppose that α_2 and α_4 are injective, and α_1 is surjective, show that α_3 is injective.
   (c) Show that if α_2, α_4 are isomorphisms, α_5 is injective and α_1 is surjective, that α_3 is also an isomorphism.