HW #4 – MATH 6320
SPRING 2015

DUE: THURSDAY FEBRUARY 26TH

(1) Determine the Galois group of $x^4 - 25$ over $\mathbb{Q}$.

(2) Let K be a field of characteristic $\neq 2$. Suppose that $\alpha, \beta \in K$. Figure out exactly when $K(\sqrt{\alpha}) = K(\sqrt{\beta})$. Use this to determine whether or not $\mathbb{Q}(\sqrt{1-\sqrt{2}}) = \mathbb{Q}(i, \sqrt{2})$.

(3) Let $K = \mathbb{Q}(a^{1/n})$ where $a \in \mathbb{Q}_{>0}$ and that $x^n - a$ is irreducible so that $[K : \mathbb{Q}] = n$. Suppose that E is a subfield of K with $[E : \mathbb{Q}] = d$. Prove that $E = \mathbb{Q}(a^{1/d})$.

   Hint: Consider $N_{K/E}(a^{1/n}) \in E$ (remember, $N_{K/E}(a^{1/n})$ was defined in the previous homework).

(4) Suppose that $m, n > 0$ are integers. What is $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z})$? Of course your answer will depend on m and n.

(5) Let A be a ring, M and A-module and I an ideal. Show that $M \otimes_A (A/I) \cong M/IM$.

(6) Suppose that R is a local ring and M, N are finitely generated R-modules. Prove that if $M \otimes_R N = 0$ then $M = 0$ or $N = 0$. Find counter examples to this statement if R is nonlocal or if M, N are not finitely generated.

   Hint: Use Nakayama's lemma.

(7) Let R be a ring $\neq 0$ with $R^n \cong R^m$ for some integers $m, n > 0$. Show that $m = n$.

   Hint: Use Nakayama's lemma.

(8) Suppose that R is a ring and that L, M, N are R-modules. Show that $\mathrm{Hom}_R(L \otimes M, N) \cong \mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N))$. Additionally, choose one of the modules L, M, N and show that this isomorphism is functorial in that variable. In other words, if you choose L and if $L \longrightarrow L'$ is a module map, show that

$$\begin{array}{ccc} \mathrm{Hom}_R(L' \otimes_R M, N) & \longrightarrow & \mathrm{Hom}_R(L \otimes_R M, N) \\ \updownarrow \sim & & \updownarrow \sim \\ \mathrm{Hom}_R(L', \mathrm{Hom}_R(M, N)) & \longrightarrow & \mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N)) \end{array}$$

commutes where the vertical maps are the isomorphisms from earlier in this problem and the horizontal maps are the ones coming from the contravariant nature of Hom.

   Hint: This is easier than you might think, try writing down where something has to go.