NOTES – MATH 6320
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As before, all rings are commutative.

1.1. Tensor products.

There is one really useful fact about localization of modules.

Lemma 1.1. *Suppose that $\phi : M \longrightarrow N$ is an injective map of R-modules and $W \subseteq R$ is a multiplicative system, then the induced map*

$$\phi' : W^{-1}M \longrightarrow W^{-1}N$$

is also injective. Equivalently the induced map, $W^{-1}R \otimes_R M \longrightarrow W^{-1}R \otimes_R N$ is injective.

Proof. Ok, what do I mean by ϕ'? $\phi'(m/w) = \phi(m)/w$ (what else could it be?) Suppose that $\phi'(m/w) = \phi(m)/w = 0$. Hence there exists $v \in W$ such that $v\phi(m) = 0$. But $v\phi(m) = \phi(vm)$ so that $vm = 0$ since ϕ is injective. But then $0 = m/v \in W^{-1}M$. $\square$

It is actually really uncommon that tensoring preserves injectivity (as we'll see in the next section). Modules L such that if $M \longrightarrow N$ is injective, then so is $L \otimes M \longrightarrow L \otimes N$ are called *flat*. Thus $W^{-1}R$ is a flat R-module.

What we have just done is a great example of a special type of tensor product called *extension of scalars*. Suppose M is an R-module, $R \longrightarrow S$ is a ring homomorphism, and we really want to make M into an S-module. The most obvious thing to do is $M \otimes_R S$. Then S can act on this tensor product on the right. For example, $\mathbb{R}[x] \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}[x]$. Likewise $\mathbb{Z}[x] \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})[x]$.

1.2. Exactness of tensor products and the Hom functor.

We have just seen that localization of modules (ie tensoring with the localized ring) preserve injectivities of modules. This is *NOT* true for arbitrary tensor products.

Example 1.2. Indeed, consider the injection $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$ and let us tensor it with $\otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$. Then we have the map

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{(\times 2)\otimes(\mathrm{id})} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}. \tag{1.2.1}$$

Note that first $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$. Let's convince ourselves of this explicitly, indeed each $a \otimes b = 1 \otimes ab$ and so we can represent each element of the tensor as an element of $\mathbb{Z}/2\mathbb{Z}$. Of course, there is a surjective map $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$ (coming from the universal property of the tensor product) and the isomorphism follows.

We return to the map and observe that $1 \otimes 1$ is sent to $2 \otimes 1 = 1 \otimes 2 = 1 \otimes 0 = 0$. In particular, the map from (1.2.1) is the zero map and hence not injective.

Tensor products do preserve a lot of other properties though.

Definition 1.3 (Short exact sequences)**.** Suppose that L, M, N are R-modules. A *short exact sequence*, denoted

$$0 \longrightarrow L \xrightarrow{\phi} M \xrightarrow{\psi} N \longrightarrow 0$$

is a pair of maps $\phi : L \longrightarrow M$ and $\psi : M \longrightarrow N$ such that ϕ is injective, ψ is surjective and $\ker \psi = \operatorname{im} \phi$.

For example, $0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ is a short exact sequence.

Example 1.4. The canonical example of a short exact sequence comes from picking $I \subseteq R$ an ideal and forming:

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

Short exact sequences are special cases of exact sequences.

Definition 1.5 (Exact sequence)**.** Suppose that $\{C_i\}$ is a collection of R-modules with maps $C_i \xrightarrow{\phi_i} C_{i+1}$, written diagrammatically as:

$$\dots \xrightarrow{\phi_{i-2}} C_{i-1} \xrightarrow{\phi_{i-1}} C_i \xrightarrow{\phi_i} C_{i+1} \xrightarrow{\phi_{i+1}} C_{i+2} \xrightarrow{\phi_{i+2}} \dots$$

This is called a (cochain) complex if $\ker \phi_i \supseteq \operatorname{im} \phi_{i-1}$ for all i. It is called an *exact sequence* if $\ker \phi_i = \operatorname{im} \phi_{i-1}$ for all i.

As we have already seen, tensor products do not preserve exact sequences (since they don't preserve injections, which can be written as exact sequences $0 \longrightarrow M \longrightarrow N$). However, the following is true.

Proposition 1.6. *If* $0 \longrightarrow L \xrightarrow{a} M \xrightarrow{b} N \longrightarrow 0$ *is an exact sequence and* T *is another* R*-module, then*

$$L \otimes_R T \xrightarrow{\alpha} M \otimes_R T \xrightarrow{\beta} N \otimes_R T \longrightarrow 0$$

is also exact.

This proposition asserts that $\otimes$ is *right-exact* (it takes short exact sequences to sequences that are exact on the right).

Proof. It is easy to see that β is surjective, indeed if $n \otimes t \in N \otimes T$, then since $M \longrightarrow N$ is surjective, there exists $m \in M$ such that $b(m) = n$. Hence $m \otimes t \mapsto n \otimes t$ and it follows that β surjects.

We now need to show that $\ker\beta = \operatorname{im}\alpha$. Let $C = \operatorname{im}\alpha$, we already know that $C \subseteq \ker\beta$ and so we have a map $\gamma : (M \otimes_R T)/C \longrightarrow N \otimes_R T$. It is sufficient to show that this map is injective. Define a map

$$\sigma : N \otimes_R T \longrightarrow (M \otimes_R T)/C$$

by $n \otimes t \mapsto \overline{b_1(n) \otimes t}$ where $b_1(n)$ is *any* $m \in M$ with $b(m) = n$ and $\overline{\bullet}$ denotes the image after modding out by C. We need to show that σ is well defined. Suppose that m and m' are such that $b(m) = b(m') = n$ then we need to show that $\overline{m \otimes t} = \overline{m' \otimes t}$ (this is the same as showing that the obvious bi-linear map from the universal property is well defined). But since $b(m) = b(m')$, there exists $l \in L$ such that $a(l) = m - m'$. Therefore since $a(l) \otimes t \in C$, we see that $\overline{(m - m') \otimes t} = 0$ and $\overline{m \otimes t} = \overline{m' \otimes t}$. This shows σ is well defined. Now, $(M \otimes_R T)/C \xrightarrow{\gamma} N \otimes_R T \xrightarrow{\sigma} (M \otimes_R T)/C$ sends $\overline{m \otimes t}$ back to itself. It follows that γ is injective. □

We'll see another proof later once we understand the relation of $\otimes$ with Hom. Indeed, at least as fundamental as the $\otimes$ functor is the Hom functor. Suppose that M, N are R-modules. Then $\operatorname{Hom}_R(M, N)$ is the set of R-module homomorphisms $M \longrightarrow N$. It is an R-module since $r.\phi$ is defined by $(r.\phi)(m) = r\phi(m) = \phi(rm)$. In other words, r can act on either the domain or the codomain, it doesn't matter. Now suppose that $\eta : L \longrightarrow M$ is a module homomorphism. Then we have an induced R-module homomorphism:

$$\Phi : \operatorname{Hom}_R(M, N) \longrightarrow \operatorname{Hom}_R(L, N)$$

defined by $\big(\Phi(f)\big)(l) = f(\eta(l))$.

On the other and, of $\delta : N \longrightarrow O$ is an R-module homomorphism, then obtain:

$$\Psi : \operatorname{Hom}_R(M, N) \longrightarrow \operatorname{Hom}_R(M, O)$$

which is defined by $\big(\Phi(f)\big)(m) = \delta(f(m))$.

Proposition 1.7. *The functors* $\operatorname{Hom}_R(\bullet, N)$ *and* $\operatorname{Hom}_R(M, \bullet)$ *are both left-exact. In other words, if*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is an exact sequence of R-modules, then

$$0 \longrightarrow \operatorname{Hom}_R(C, N) \xrightarrow{g'} \operatorname{Hom}_R(B, N) \xrightarrow{f'} \operatorname{Hom}_R(A, N)$$

is exact and

$$0 \longrightarrow \operatorname{Hom}_R(M, A) \xrightarrow{f''} \operatorname{Hom}_R(M, B) \xrightarrow{g''} \operatorname{Hom}_R(M, C)$$

is also exact.

Proof. See the worksheet. □