NOTES – MATH 6320
FALL 2017

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Throughout today, all rings are commutative.

1.1. **Tensor products.**

We begin by introducing tensor products. Suppose that R is a ring and that M and N are R-modules.

Suppose we wish to multiply elements of m and n, formally, and consider the resulting as an R-module. The tensor product lets us do exactly that. In particular, the tensor product $M \otimes_R N$ is generated by elements $m \otimes n$. Note that in order for it to be a module, it has to be closed under addition, and so we

(i) have to allow finite sums $\sum_{i=1}^t m_i \otimes n_i$.

We also want our multiplication to be distributive, and so we must have

(ii) $(m + m') \otimes n = m \otimes n + m' \otimes n$ and $m \otimes (n + n') = m \otimes n + m \otimes n'$.

Finally, we need to describe our action of R on this product. We have

(iii) $(rm) \otimes n = m \otimes (rn) = r.(m \otimes n)$. In other words, only elements of R can move over the tensor product.

Elements of r of course must also distribute across sums:

(iv) $r. \sum_{i=1}^t m_i \otimes n_i = \sum_{i=1}^t (rm_i) \otimes n_i$

Formally, the tensor product $M \otimes_R N$ is the free Abelian group generated by all ordered pairs $m \otimes n := (m, n) \in M \times N$ modulo the relations generated by properties (ii), (iii) and (iv).

Proposition 1.1 (Universal property of the tensor product)**.** *If $f : M \oplus N \longrightarrow L$ is a bilinear map of R-modules, then then there exists a unique R-linear $\phi : M \otimes N \longrightarrow L$ such that $\phi(m \otimes n) = f(m, n)$. Note that the obvious map $M \oplus N \longrightarrow M \otimes N$ is bi-linear.*

Now suppose that $N = S$ is an R-algebra (a ring with map $R \longrightarrow S$). Then we will frequently form the tensor product $M \otimes_R S$. This is both an R-module and an S-module (S acts on S and extends linearly).

Definition 1.2 (Localization of a module)**.** Suppose now that R is a ring, W is a multiplicative system and M is an R-module. Then the localization $W^{-1}M$ is the set of pairs $(m, w) \in M \times W$ modulo the equivalence relation

$(m, w) \sim (m', w')$ if there exists $v \in W$ such that $vw'm = vwm'$. Equivalence classes $[(m, w)]$ are denoted by m/w. $W^{-1}M$ becomes a $W^{-1}R$-module with the following addition and $W^{-1}R$-action.

$$\begin{array}{rcl} m/w + m'/w' & = & \frac{w'm+wm'}{ww'} \\ (r/w).(m/w') & = & rm/(ww') \end{array}$$

Proposition 1.3. *Suppose R is a ring, M is an R-module and W is a multiplicative system. Then:*

$$W^{-1}R \otimes_R M \cong W^{-1}M.$$

even as $W^{-1}R$-modules.

Proof. The tensor product $W^{-1}R \otimes_R M$ is very simple as tensor products go. Indeed, notice that

$$\begin{array}{rl} & (r/w \otimes m) + (r'/w' \otimes m') \\ = & (\frac{rw'}{ww'} \otimes m) + (\frac{r'w}{ww'} \otimes m') \\ = & (\frac{1}{ww'} \otimes (rw'm)) + (\frac{1}{ww'} \otimes (r'wm')) \\ = & \frac{1}{ww'} \otimes (rw'm + r'wm'). \end{array}$$

It follows that every element of $W^{-1}R \otimes_R M$ can be expressed as $\frac{1}{w} \otimes m$. Since it is easy to see that the map $W^{-1}R \oplus M \longrightarrow W^{-1}M$, $(r/w, m) \mapsto rm/w$ is bilinear, by the universal property of the tensor product, we have a map

$$\phi : W^{-1}R \otimes M \longrightarrow W^{-1}M.$$

We need to show it is an isomorphism. Certainly it is surjective, so now choose $\frac{1}{w} \otimes m \in W^{-1}R$ and suppose that $\phi(\frac{1}{w} \otimes m) = m/w = 0$. Hence there exists $v \in W$ such that $vm = 0$. But then

$$\frac{1}{w} \otimes m = \frac{v}{wv} \otimes m = \frac{1}{wv} \otimes vm = \frac{1}{wv} \otimes 0 = 0.$$

Checking that the map is a $W^{-1}R$-module homomorphism is routine and will be left to the reader. □

There is one really useful fact about localization of modules.

Lemma 1.4. *Suppose that $\phi : M \longrightarrow N$ is an injective map of R-modules and $W \subseteq R$ is a multiplicative system, then the induced map*

$$\phi' : W^{-1}M \longrightarrow W^{-1}N$$

is also injective. Equivalently the induced map, $W^{-1}R \otimes_R M \longrightarrow W^{-1}R \otimes_R N$ is injective.

We'll prove this next time.