HW #8 – MATH 6310
FALL 2017

DUE: FRIDAY, NOVEMBER 17TH

(1) Suppose that R is a commutative ring and $\phi : M \longrightarrow N$ is a homomorphism of R-modules. Suppose that L is another R module. Show that there is an R-module homomorphism
$$\psi : \mathrm{Hom}_R(N, L) \longrightarrow \mathrm{Hom}_R(M, L).$$
Further show that if ϕ is surjective, then ψ is injective.

(2) Let M be a left $\mathbb{Q}$-module. If we fix the Abelian group structure $+$ on M, show that there is only $\mathbb{Q}$-module structure that can be put on M.

(3) Suppose that R is commutative and that $\phi : R^{\oplus n} \longrightarrow R^{\oplus n}$ is a surjective R-module homomorphism. Show that ϕ is bijective. Give an example to show that ϕ can be injective without being bijective.

(4) Show that if p is prime and $e > 0$ then $\mathbb{Z}/(p^e)$, viewed as a $\mathbb{Z}$-module, cannot be written as a direct sum of two nonzero $\mathbb{Z}$-modules. On the other hand, if p and q are relatively prime integers, show that
$$\mathbb{Z}/(pq) \cong \mathbb{Z}/(p) \oplus \mathbb{Z}/(q)$$
as $\mathbb{Z}$-modules.

(5) Let M and N be R-modules and suppose that $f : M \longrightarrow N$ and $g : N \longrightarrow M$ are R-module homomorphisms such that $f \circ g = \mathrm{Id}_N$. Show that $M \cong \ker(f) \oplus \mathrm{image}(g)$.

(6) Obtain a *normal form* for the $\mathbb{Z}$-matrix
$$\begin{bmatrix} 6 & 2 & 3 & 0 \\ 2 & 3 & -4 & 1 \\ -3 & 3 & 1 & 2 \\ -1 & 2 & -3 & 5 \end{bmatrix}$$

(7) Suppose that D is a Euclidean domain. Show that any invertible matrix is a product of elementary matrices.