HW #3 – MATH 6310
FALL 2017

DUE: FRIDAY, SEPTEMBER 15TH

(1) Let G be a finite group and $\alpha : G \longrightarrow G$ be an automorphism. Set

$$I = \{g \in G \mid \alpha(g) = g^{-1}\}.$$

If $|I| > \frac{3}{4}|G|$ show that G is Abelian.

(2) A *monic* homomorphism of groups is a homomorphism $f : B \longrightarrow C$ of groups satisfying the following property. If we have any two group homomorphisms $g, h : A \longrightarrow B$ such that $f \circ g = f \circ h$, then $g = h$.

Prove that f is monic if and only if it is injective.

(3) Suppose $S \subseteq G$ is a subset of a group such that $gSg^{-1} \subseteq S$ for all $g \in G$. Show that $\langle S \rangle$ is normal. Next let $T \subseteq G$ be another subset and form the set $V = \bigcup_{g \in G} g^{-1}Tg$. Show that $\langle V \rangle$ is the unique smallest normal subgroup of G containing T.

(4) Suppose that $H \leq G$ is a subgroup of a finite group and that $n = [G : H]$. Show that H contains a normal subgroup K of G where $[G : K]$ divides $n!$.

(5) Let p be the smallest prime dividing the order of a finite group. Show that any subgroup $H \leq G$ of index p is normal.

(6) Consider the free group F on the set $\{x_2, \ldots, x_n\}$. Consider the subgroup K generated by the following elements $x_i^2, (x_i x_j)^3, (x_i x_j x_i x_k)^2$ (for i, j, k distinct). Show that F/K is isomorphic to S_n.

Hint: First note that $\{(12), (13), \ldots, (1n)\}$ generate S_n. Consider the map from $F \longrightarrow S_n$ sending $x_i \mapsto (1i)$.