WORKSHEET #5 – MATH 6140
SPRING 2019

DUE WEDNESDAY, APRIL 3RD

You may work in groups of up to 3. Only one worksheet needs to be turned in per group.

The following exercises are essentially Hartshorne, III, Exercise 2.3.

Suppose that X is a topological space and $Z \subseteq X$ is closed. Consider the functor $\Gamma_Z(X, \bullet)$ which is defined by $\Gamma_Z(X, \mathcal{F}) = \{s \in \Gamma(X, \mathcal{F}) \mid \mathrm{Supp}\,(s) \subseteq Y\}$.[1] It is straightforward to see that this is a left exact functor. The right derived functors are denoted by $H^i_Z(X, \bullet)$, and by the usual arguments, given a short exact sequence of Abelian groups,

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

we obtain a long exact sequence

$$0 \longrightarrow H^0_Z(X, \mathcal{E}) \longrightarrow H^0_Z(X, \mathcal{F}) \longrightarrow H^0_Z(X, \mathcal{G}) \longrightarrow H^1_Z(X, \mathcal{E}) \longrightarrow H^1_Z(X, \mathcal{F}) \longrightarrow H^1_Z(X, \mathcal{G}) \longrightarrow \dots$$

1. Show that if $\mathcal{E}$ is flasque, then $H^i_Z(X, \mathcal{E}) = 0$ for $i > 0$.

Hint: Show that if $\mathcal{E}$ is flasque in the above short exact sequence, then $H^0_Z(X, \mathcal{F}) \twoheadrightarrow H^0_Z(X, \mathcal{G})$ surjects.

[1] Recall that $\mathrm{Supp}\,(s) = \{x \in X \mid 0 \neq s_x \in \mathcal{F}_x\}$.

2. Let $U = X \setminus Z$. Show that for any $\mathcal{F}$, there is a long exact sequence:

$$0 \longrightarrow H^0_Z(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(U, \mathcal{F}) \longrightarrow H^1_Z(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(U, \mathcal{F}) \longrightarrow \dots$$

Hint: See worksheet 1, exercise #8, and show that sequence from that exercise is exact if $\mathcal{F}$ is flasque. Also feel free to use anything from Chapter III, Section 1 of Hartshorne (or other sources for formal nonsense).

3. Suppose that A is a ring and that $I \subseteq A$ is a finitely generated ideal. Consider the functor from A-modules to A-modules defined by $\Gamma_I(M) = \{m \in M \mid I^n m = 0 \text{ for some } n > 0.\}$. Show that this functor is left exact. Its right derived functors are denoted $H^i_I(\bullet)$ and are called local cohomology.

4. Suppose that A is a Noetherian ring, $I \subseteq A$ is an ideal, and M is an A-module. Let $X = \operatorname{Spec} A$ and $Z = V(I) \subseteq X$. Prove that

$$H^i_I(M) = H^i_Z(X, \widetilde{M})$$

for all $i > 0$.

What follows are essentially exercises from Hartshorne chapter III section 3.

Suppose that A is a ring, M is a finitely generated A-module and $I \subseteq A$ is an ideal. Further suppose that $M \neq IM$. Recall that the *I-depth of M, or the depth*[2] *of M along I* is the maximum length of a sequence of elements $\{x_1, \ldots, x_d\} \subseteq I$ so that x_{i+1} is not a zero divisor on $M/(x_1, \ldots, x_i)M$. This is denoted by $\operatorname{depth}_I M$.

5. For A, I, M as above, show that $\operatorname{depth}_I M \geq k$ if and only if $H^i_I(M) = 0$ for all $i < k$.

Hint: Use induction, the base case $k = 1$ is straightforward.

[2]Some CA sources call this grade, Hartshorne calls it depth, as does the stacks project. I'll stick with their terminology. In the case of a local ring with $I = \mathfrak{m}$, these notions agree.

Suppose $(X, \mathcal{O}_X)$ is a scheme. A coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is said to satisfy *Serre's* S_k *property* if for each point $p \in X$ (including non-closed points), we have that $\operatorname{depth}_{\mathfrak{m}_p} \mathcal{F}_p \geq \min(\dim \mathcal{O}_{X,p}, k)$. Likewise, a finitely generated module M over a ring A is S_k if for each $Q \in \operatorname{Spec} A$, we have $\operatorname{depth}_Q M_Q \geq \min(\dim A_Q, k)$, this is equivalent to $\widetilde{M}$ satisfying S_k in the above sense. [3]

6. Suppose X is a scheme and $Z \subseteq X$ is a closed subscheme of codimension[4] $\geq k$. Show that if $\mathcal{F}$ is S_k, then $H^i_Z(X, \mathcal{F}) = 0$ for all $i < k$. If you just want to do the case where $k = 2$, that is fine too, and somewhat easier.

Hint: Fix k and use induction on i. Fix a point $p \in Z$ (maybe even a generic point of Z) and consider the long exact sequence you get by from $H^i_Z(X, \bullet)$. On the other hand, you can apply $H^i_p(X, \bullet)$ to this long exact sequence (or really to the short exact sequences you can break up the long exact sequence into). This is a bit easier if you know spectral sequences, in which case just consider the spectral sequence obtained by composing the derived functors $H^j_p(X, \bullet)$ and $H^i_Z(X, \bullet)$.

[3]Warning, in some texts, $\dim \mathcal{O}_{X,p}$ or $\dim A_Q$ is replaced by $\dim \mathcal{F}_p$ or $\dim M_Q$.

[4]Codimension $\geq k$ means for every irreducible component Z_i of Z, there exists a chain of irreducible subsets $Z_i = Y_0 \subseteq Y_1 \subseteq \ldots Y_k \subseteq X$.

7. Conclude that if $\mathcal{F}$ is S_2 and $Z \subseteq X$ is codimension 2, then $\Gamma(X, \mathcal{F}) = \Gamma(X \setminus Z, \mathcal{F})$.

8. Suppose again that X is a scheme, that $\mathcal{F}$ is S_2 and $Z \subseteq X$ is codimension 2. Let $U = X \setminus Z$ and suppose that $i : U \longrightarrow X$ is the (open) inclusion. Show that $i_* \mathcal{F}|_U \cong \mathcal{F}$.