WORKSHEET #4 – MATH 6140
SPRING 2019

DUE FRIDAY, MARCH 8TH

You may work in groups of up to 3. Only one worksheet needs to be turned in per group.

We begin with some definitions.

Definition: *A Cartier divisor D on a normal projective variety X is called:*

(a) very ample *if $|D|$ induces an embedding into projective space.*
(b) ample *if nD is very ample for some $n > 0$.*
(c) semi-ample *if $|nD|$ is base point free for some $n > 0$.*
(d) big *if the induced rational map to projective space $\phi_{|nD|} : X \dashrightarrow \mathbb{P}^m$ is birational onto its image.*

The more standard definition of ample is the following:

(b') *A Cartier divisor D is* ample *if for any coherent sheaf $\mathcal{F}$, there exists an integer n_0 so that $\mathcal{F} \otimes \mathcal{O}_X(nD) =: \mathcal{F}(nD)$ is globally generated for all $n \geq n_0$.*

These are indeed equivalent, as we'll show shortly.

1. Suppose that D is very ample and that E is such that $|E|$ is base point free. Prove that $D + E$ is also very ample. In particular, conclude that if D is very ample, so is $2D$, $3D$, etc, and likewise with ample.

Hint: You have maps $\phi_{|D|} : X \longrightarrow \mathbb{P}^r$ and $\phi_{|E|} : X \longrightarrow \mathbb{P}^s$. Consider the induced product map $X \longrightarrow \mathbb{P}^r \times \mathbb{P}^s$. Show that this corresponds to a linear system that is contained within the linear system $|D + E|$.

2. Suppose that D is very ample. Prove that D satisfies the condition (b') above. Conclude that ample divisors in the sense of (b) are ample in the sense of (b').

Hint: Suppose that $i : X \longrightarrow \mathbb{P}^n$ is the induced closed embedding. Prove that $\Gamma(X, \mathcal{F}(nD)) = \Gamma(\mathbb{P}^n, (i_*\mathcal{F}) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(n))$.

3. Suppose that D is a semi-ample divisor on X. Further suppose that there are two points p and q so that if $D' \sim nD$ and p or q are in the support of D', then p and q are in the support of D'. Prove that D cannot be ample.

Hint Use Lemma 13.28 from the text.

The following can also be found in Cutkosky/Hartshorne. Suppose $\phi : X \longrightarrow \mathbb{P}^n$ corresponds to a vector space $V \subseteq \Gamma(X, \mathcal{O}_X(D))$ where D is a Cartier divisor.

(i) ϕ is regular at a point p if and only if the canonical map $V \longrightarrow \mathcal{O}_X(D)/I_p(D) \cong k$ surjects. (This map is just the restriction map, followed by quotienting).

(ii) ϕ is injective and regular if and only if for every points $p \neq q \in X$, we have that the natural map $V \longrightarrow (\mathcal{O}_X(D)/I_p(D)) \oplus (\mathcal{O}_X(D)/I_q(D)) \cong k^2$ surjects.

(iii) ϕ is a closed embedding if and only if it is injective and regular and for each $p \in X$, we have that the set $\{s_p \in \mathfrak{m}_p\mathcal{O}_X(D)_p | s \in V\}$ spans the dual tangent space $(\mathfrak{m}_p\mathcal{O}_X(D))/(\mathfrak{m}_p^2\mathcal{O}_X(D)) \cong \mathfrak{m}_p/\mathfrak{m}_p^2$.

Condition (ii) is usually called $|V|$ separating points and (iii) is called $|V|$ separating tangent vectors. We've basically done (i) and (ii), we will work on showing (iii) now.

4. Consider $V = \Gamma(X, \mathcal{O}_X(D))$. Prove that the condition from (iii) holds if D is very ample in the sense of (b).

Hint: Observe that $\mathcal{O}_{\mathbb{P}^n}(1)$ satisfies the condition in (iii).

5. Prove that the condition from (iii) really does imply ϕ is a closed embedding.

Hint: You may use the following algebra fact. If $f : R \longrightarrow S$ is a local map of local rings such that S is a finite R-module, $R/\mathfrak{m}_R \longrightarrow S/\mathfrak{m}_S$ is an isomorphism, and $\mathfrak{m}_R/\mathfrak{m}_R^2 \longrightarrow \mathfrak{m}_S/\mathfrak{m}_S^2$ is surjective, then f is surjective.

6. Show that (b') implies (b) and thus our different characterizations of ampleness all coincide.

7. Suppose that $i : Y \hookrightarrow X$ is a closed embedding of projective varieties and that $D \subseteq X$ is an ample (resp. semi-ample) divisor such that $Y \not\subseteq \operatorname{Supp} D$. Prove that $i^*D = D|_Y$ is ample (resp. semi-ample).

8. Give an example of a projective variety X, a closed subvariety $Y \subseteq X$, and a big divisor D on X such that $Y \not\subseteq \operatorname{Supp} D$, so that $D|_Y$ is not big.

8. Give an example of a surjective map of projective varieties (of dimension > 0) $\phi : Y \longrightarrow X$ and a big divisor D on X such that ϕ^*D is not big.

Hint: If D is big, then the map associated to $|nD|$ is injective at most points.