WORKSHEET #3 – MATH 6140
SPRING 2019

DUE MONDAY, FEBRUARY 18TH

You may work in groups of up to 3. Only one worksheet needs to be turned in per group.

We begin by recording some definitions (we'll talk about these soon).

Definition. We work on a normal quasi-projective variety X and we fix $\mathcal{K}(X)$ to be the constant sheaf made out of the function field $K(X)$ of X. Suppose D is a Weil divisor. Set $\mathcal{O}_X(D) \subseteq \mathcal{K}(X)$ denote the sheaf of $\mathcal{O}_X$-modules with the property that

$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in K(X) \mid \mathrm{Div}_U(f) + D \geq 0\}.$$

A divisor D is called a *Cartier divisor* if $\mathcal{O}_X(D)$ is an invertible sheaf. In other words, for each $p \in X$, we have isomorphisms of stalks $\mathcal{O}_X(D)_p = \mathcal{O}_{X,p}$.

1. Show that every principal divisor on a non-singular variety is Cartier.

2. Suppose that D is a Cartier divisor and E is a Weil divisor. Prove that

$$\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E) \cong \mathcal{O}_X(D) \cdot \mathcal{O}_X(E) = \mathcal{O}_X(D+E) \subseteq \mathcal{K}(X).$$

3. Consider $X = Z(xy - z^2) \subseteq \mathbb{A}^3$. Verify that the ideal $\langle x, z \rangle \subseteq k[x, y, z]/\langle xy - z^2 \rangle = k[X]$ is prime of height one and so determines a prime divisor E. Show that E is not Cartier but that $2E$ is.

Suppose that $\pi : Y \longrightarrow X$ is a finite surjective morphism of normal affine varieties (so that $\pi^* : k[X] \longrightarrow k[Y]$ makes $k[Y]$ into a finite $k[X]$-modules). For every prime divisor E on X, with corresponding prime ideal I_E and discrete valuation v_E, there exist finitely many prime divisors $F_1, \ldots, F_n \subseteq Y$ that map onto E. Suppose that $e \in \mathcal{O}_{X,E}$ is an element with $v_E(e) = 1$. Write $a_i = v_{F_i}(e)$. We write

$$\pi^* E = \sum_i a_i F_i.$$

For more general divisors $\sum b_j E_j$, we extend by linearity.

4. Consider the map $\pi : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ defined by $\pi(t) = t^2$. Compute $\pi^*(2 \cdot Z(x) - 3 \cdot Z(x-1) + 1 \cdot Z(x+1))$.

5. Suppose that $\pi : Y \longrightarrow X$ is a finite surjective morphism of normal affine varieties and that $f \in K(X)$. Show that

$$\pi^* \mathrm{Div}_X(f) = \mathrm{Div}_Y(f).$$

Suppose that $\pi : Y \longrightarrow X$ is a surjective morphism of normal varieties. Suppose that D is a Cartier divisor on X. For every sufficiently small open chart U on X, since $\mathcal{O}_X(D)$ is locally free, we can write $f_i \cdot \mathcal{O}_{U_i} = \mathcal{O}_X(D)|_{U_i}$ for some $f_i \in K(X)$. We define $\pi^* D$ to be the divisor $\mathrm{Div}_{\pi^{-1}(U_i)}(f_i)$ on $\pi^{-1}(U_i)$.

6. Suppose that $\pi : Y \longrightarrow X$ is a surjective morphism of normal affine varieties and D is a Cartier divisor on X. Prove that the two definitions of $\pi^* D$ agree.

7. Consider $\pi : Y \longrightarrow X = \mathbb{A}^2$ to be the blowup of the origin. Let $D = \mathrm{Div}_X((y^2 - x^3)/y)$. Write $\pi^* D$ as a sum of prime divisors.