WORKSHEET #2 – MATH 6140
SPRING 2019

DUE FRIDAY, FEBRUARY 1ST (ELECTRONICALLY)

You may work in groups of up to 3. Only one worksheet needs to be turned in per group.

1. Suppose that $\phi : Y \longrightarrow X$ is a regular map of quasi-projective varieties and that $\mathcal{E}$ a locally free sheaf of $\mathcal{O}_X$-modules. Prove that $\phi^*\mathcal{E}$ is also locally free as a sheaf of $\mathcal{O}_Y$-modules.

2. Suppose that $\phi : Y \longrightarrow X$ is a regular map of quasi-projective varieties. Suppose that $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_X$-modules on X. Further suppose that $\mathcal{F}$ is a sheaf of $\mathcal{O}_Y$-modules. Prove that there is a natural isomorphism of $\mathcal{O}_Y$-modules

$$(\phi_* \mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{E} \cong \phi_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \phi^* \mathcal{E}).$$

(don't actually prove the "natural" part, but at least state what you would have to show)

3. Let X be a quasi-projective variety and suppose that $\mathcal{F}$ is a coherent sheaf on X. Suppose that there is another coherent sheaf $\mathcal{G}$ on X such that

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \cong \mathcal{O}_X.$$

Prove that $\mathcal{F}$ is locally free and of rank 1.

Hint: Nakayama's lemma and the ideas surrounding it may help.

4. Suppose that X is a quasi-projective variety and that $\mathcal{F}$ is a locally free sheaf of rank 1 on X. Prove that there exists a coherent sheaf $\mathcal{G}$ so that

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \cong \mathcal{O}_X.$$

In view of this problem and its predecessor, locally free sheaves of rank 1 are called *invertible sheaves.*

5. Suppose X is an affine variety and that $R = k[X]$. Suppose that M is an R-module. For any $m \in M = \Gamma(X, \widetilde{M})$, show that $\operatorname{Supp} m = Z(\operatorname{Ann}_R m)$. Here $\operatorname{Ann}_R m = 0 :_R m = \{r \in R \mid rm = 0\}$ and $\operatorname{Supp} m$ is defined as in the previous worksheet.

6. With notation as above, if M is finitely generated, show that $\operatorname{Supp} \widetilde{M} = Z(\operatorname{Ann}_R M)$.