WORKSHEET #1 – MATH 6140
SPRING 2019

DUE FRIDAY, JANUARY 18TH

You may work in groups of up to 3. Only one worksheet needs to be turned in per group.

1. Suppose that $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ is a map of sheaves (on X). Determine which of the following presheaves are actually sheaves. If they are, prove it. If they are not, give an example.

(a) the presheaf $\mathcal{H}$ such that $\mathcal{H}(U) = \ker\big(\mathcal{F}(U) \longrightarrow \mathcal{G}(U)\big)$.

(b) the presheaf $\mathcal{H}(U) = \operatorname{im}\big(\mathcal{F}(U) \longrightarrow \mathcal{G}(U)\big)$.

(c) the presheaf $\mathcal{H}(U) = \operatorname{coker}\big(\mathcal{F}(U) \longrightarrow \mathcal{G}(U)\big) = \mathcal{G}(U)/\mathcal{F}(U)$.

The sheafification of (a),(b),(c) (if sheafification is necessary) are denoted $\ker \phi$, $\operatorname{im} \phi$ and $\operatorname{coker} \phi$ respectively.

2. Suppose that $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ is a short exact sequence of sheaves on X (meaning it is short exact on the stalks). Prove carefully that for every open $U \subseteq X$, the sequence:

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U)$$

is exact.

3. A sheaf $\mathcal{F}$ is *flasque* if for every $V \subseteq U$ (open sets in X), the restriction map $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is surjective. With the notation as in **2.**, show that if $\mathcal{F}$ is flasque, then

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U) \longrightarrow 0$$

is exact.

4. Suppose that $f : X \longrightarrow Y$ is a map of topological spaces and $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ is a short exact sequence of sheaves on X. Prove that

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow f_*\mathcal{G} \longrightarrow f_*\mathcal{H}$$

is exact on Y. Further give an example to show that $f_*\mathcal{G} \longrightarrow f_*\mathcal{H}$ is not necessarily surjective.

5. Suppose that $\phi : X \longrightarrow Y$ is a regular map of affine varieties. Suppose that M is a $k[X]$ modules and $\mathcal{F} = \widetilde{M}$. Note that M can also be viewed as a $k[Y]$ module by the map $\phi^* : k[Y] \longrightarrow k[X]$. We denote this module by $M_{k[Y]}$ and let $\mathcal{G} = \widetilde{M_{k[Y]}}$. Show that $\phi_* \mathcal{F} \cong \mathcal{G}$.

If $\phi : X \longrightarrow Y$ is a continuous map of topological spaces and $\mathcal{G}$ is a sheaf on Y, we define $\phi^{-1}(\mathcal{G})$ to be the sheafification of the presheaf $\mathcal{H}$ where for any $U \subseteq X$, we form the direct limit:

$$\mathcal{H}(U) = \lim_{V \supseteq f(U)} \mathcal{G}(V).$$

6. Suppose that $Z \subseteq X$ is a subset of a topological space and that $\mathcal{F}$ is a sheaf on X. Let $i : Z \longrightarrow X$ be the inclusion and show that for each point $z \in Z$,

$$(i^{-1}\mathcal{F})_z = \mathcal{F}_z.$$

In the case that Z is a closed subvariety of a variety X, how do $i^{-1}\mathcal{O}_X$ and $\mathcal{O}_Z$ compare?

7. Suppose that $\phi : X \longrightarrow Y$ is a continuous map of topological spaces. Show that there is a natural[1] bijection of sets for any sheaves $\mathcal{G}$ on Y and $\mathcal{F}$ on X.

$$\mathrm{Hom}_X(\phi^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_Y(\mathcal{G}, \phi_*\mathcal{F}).$$

First note that if $\mathcal{F}$ is a sheaf, we define the *support of* $\mathcal{F}$, denoted $\mathrm{Supp}\,\mathcal{F}$, to be $\{x \in X \mid \mathcal{F}_x \neq 0\}$. If $s \in \mathcal{F}(U)$, we define $\mathrm{Supp}\,(s) = \{x \in U \mid s_x \neq 0\}$. It is not difficult to show that $\mathrm{Supp}\,(s)$ is always closed but that $\mathrm{Supp}\,\mathcal{F}$ is not always closed.

8. Suppose that $Z \subseteq X$ is a closed subset and let $\mathcal{F}$ be a sheaf on X. Consider the presheaf $\mathscr{H}^0_Z(\mathcal{F})$ defined as

$$\mathscr{H}^0_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \mathrm{Supp}\, s \subseteq Z\}.$$

Show that $\mathscr{H}^0 Z(\mathcal{F})$ is a sheaf and that if $i : (X \setminus Z) \longrightarrow X$ is the inclusion, there exists an exact sequence:

$$0 \longrightarrow \mathscr{H}^0_Z(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow i_*(\mathcal{F}|_{X\setminus Z}).$$

[1]Actually, please don't show the *natural* part, but if you don't know what *natural* means, ask.