WORKSHEET #7 – MATH 6130
FALL 2018

DUE MONDAY, NOVEMBER 12TH

You may work in groups of up to 3. Only one worksheet needs to be turned in per group.

Setup. Suppose that $X \subseteq \mathbb{A}^n$ is an affine variety and $I \subseteq k[X]$ is an ideal (cutting out an closed subset $Z(I) = Z \subseteq X$). Suppose $J \subseteq k[\mathbb{A}^m]$ is an ideal with $Y = Z(J)$. Finally suppose we have an injective[1] k-algebra map $\psi : k[\mathbb{A}^m]/J \hookrightarrow k[X]/I$ that makes $k[X]/I$ into a (usually finite) $k[\mathbb{A}^m]/J$-module.

Consider the subring of $k[X]$:

$$A := \{f \in k[X] \mid \overline{f} \in \psi(k[\mathbb{A}^m]/J) \subseteq k[X]/I\}.$$

In other words, A is elements of $k[X]$ whose image in $k[X]/I$ is contained the image of $k[\mathbb{A}^m]/J$ in $k[X]/I$.

1. Show that there is a commutative diagram of k-algebra morphisms:

$$\begin{array}{ccc} k[X] & \twoheadrightarrow & k[X]/I \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\psi} \\ A & \xrightarrow[\beta]{} & k[\mathbb{A}^m]/J. \end{array}$$

2. Show that the map β above is surjective.

[1]This isn't really necessary, but it makes life easier and the geometry simpler.

3. Show that the fraction field of A is the same as the fraction field of $k[X]$.

Hint: Show that every element of I is in A.

4. Assume ψ above is finite. Show that $k[X]$ is integral over A. Hence, if X is normal, then $k[X]$ is the normalization of A.

Hint: Since ψ is finite, every element $\overline{f}$ of $k[X]/I$ satisfies an integral expression over $k[\mathbb{A}^m]/J \cong A/(\ker \beta)$ (even though these rings are not domains),

$$\overline{f}^n + \overline{a}_{n-1}\overline{f}^{n-1} + \cdots + \overline{a}_1\overline{f}^1 + \overline{a}_0 = 0 \in k[X]/I$$

with $a_i \in A$.

Suppose $A \cong k[Y]$ for some affine variety Y. We have an induced map of varieties:

$$\begin{array}{ccc} X & \hookleftarrow & Z_X(I) \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ Y & \hookleftarrow & Z(J). \end{array}$$

where the vertical surjections are because the ring maps are finite.

5. Consider $X = \mathbb{A}^1$ with $k[X] = k[x]$. $I = (x(x-1))$. $J = (0) \subseteq k = k[]$. Compute A and Y and draw the diagram to show that $X \twoheadrightarrow Y$ is not bijective.

6. Consider $X = \mathbb{A}^1$ with $k[X] = k[x]$. $I = (x^2)$. $J = (0) \subseteq k$. Compute A and Y and show that $X \longrightarrow Y$ is bijective.

7. Show that the map $X \longrightarrow Y$ induces an isomorphism of quasi-affine varieties:

$$X \setminus Z_X(I) \longrightarrow Y \setminus \mathrm{Image}(Z(J)) = Y \setminus Z(\ker \beta).$$

Hint: Invert an element $f \in \ker \beta$ and show that $A[f^{-1}] \longrightarrow k[X][f^{-1}]$ is an isomorphism.

What we have shown so far is that Y is obtained from X excepted that we replaced $Z_X(I)$ (and possibly some tangent data around it as in **6.**) with $\mathrm{Image}(Z(J))$. We do one more example.

8. Consider $X = \mathbb{A}^2$ with $k[X] = k[x, y]$ and set $I = (y)$. Finally let $J = (0) \subseteq k[t]$ with the map $\psi : k[t] \longrightarrow k[x, y]/(y) \cong k[x]$ sending $t \mapsto x^2$. Compute A and describe it geometrically in the case that $\mathrm{char} k \neq 2$.

9. Suppose that W is a non-normal affine variety with normalization (with respect to $k(W)$), and that $X \longrightarrow W$ is the normalization (so that $k[W] \hookrightarrow k[X]$ is a finite map, both rings in $k(W)$). Let $\mathfrak{c} = \{f \in k[W] \mid f \cdot k[X] \subseteq k[W]\}$. Show that $\mathfrak{c}$ is an ideal in both $k[W]$ and $k[X]$. It is called the *conductor of* $X \longrightarrow W$ *(or of* $k[W] \subseteq k[X]$*).*

10. If we set $I = \mathfrak{c} \subseteq k[X]$ and write $k[\mathbb{A}^m]/J = k[W]/\mathfrak{c}$ with $\psi : k[W]/\mathfrak{c} \longrightarrow k[X]/\mathfrak{c}$ the canonical map, show that the associated A is isomorphic to $k[W]$. In particular, *every* non-normal affine variety is constructed by replacing a subvariety (and possibly some tangent data) of its normalization with another variety (with tangent data).

Extra. You might find it amusing to compute A in the case of $X = \mathbb{A}^2$, $I = (x)$ and $J = (0) \subseteq k$.