WORKSHEET #6 - MATH 6130 FALL 2018

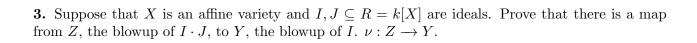
DUE WEDNESDAY, OCTOBER 31ST

You may work in groups of up to 3. Only one worksheet needs to be turned in per group.

1. Suppose that X is an affine variety with R = k[X]. Consider an ideal $I \subseteq R$. Show that the blowup of I is isomorphic to the blowup of I^2 .

Hint: The blowup lives in $X \times \mathbb{P}^n$ where $I = (f_0, \dots, f_n)$. I^2 is generated by $n^2 + 2n + 1$ elements $(f_0^2, f_0 \cdot f_1, \dots, f_n^2)$. Note on the other hand we have the Segre embedding $\mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n \subseteq P^N$ where $N = n^2 + 2n$.

2. Consider $X = \mathbb{A}^2$ with k[X] = k[x,y]. Show that there is a map from the blowup of (x^2, xy, y^2) (which is isomorphic to the blowup of (x,y) by **1.**), to the blowup of (x^2,y^2) . Is it an isomorphism?



4. With notation as in **3.**, after picking generators f_i for I with corresponding affine charts $U_i \subseteq Y$, show $\nu^{-1}(U_i) \to U_i$ is the blowup of the extended ideal $J \cdot k[U_i]$ (extended via the map $k[X] \to k[U_i]$).

5. Consider $X = Z(x \cdot y - z^2) \subseteq \mathbb{A}^3$. Consider $\pi : Y \to X$ the blowup of the ideal I = (x, z). Find the locus on which π is an isomorphism and show it is strictly larger than $X \setminus Z_X(I)$.

6. Suppose that $X \subseteq Y$ is a closed subvariety of an affine variety. Consider an ideal $I \subseteq k[x_1,\ldots,x_n]=k[\mathbb{A}^n]$ and let $\pi:\widetilde{Y}\to Y$ be the blowup of I. Let \widetilde{X} be the strict transform of X in Y. Prove (without looking *too* closely at the book, which also proves this) that $\pi|_{\widetilde{X}}:\widetilde{X}\to X$ is the blowup of $I\cdot k[X]$.

- 7. Suppose that I is a homogenous prime ideal in $k[x_0,\ldots,x_n]=R$ not containing $\mathfrak{m}=(x_0,\ldots,x_n)$. Consider the associated variety $X\subseteq\mathbb{A}^{n+1}$ (this is called the affine cone over the projective variety $Z=Z_{\mathbb{P}^n}(I)\subseteq\mathbb{P}^n$ try drawing a picture, because I obviously can't). Consider the blowup of \mathfrak{m} , $\pi:\widetilde{Y}\to\mathbb{A}^{n+1}$ and let $E\cong\mathbb{P}^n=\pi^{-1}(\text{origin})$ be the exceptional set¹. Further let $\widetilde{X}\subseteq\widetilde{Y}$ be the strict transform of tldX.
- (a) If I=(f) is a principal ideal, consider the affine charts $U_0,\ldots,U_n\subseteq\widetilde{Y}$ corresponding the generators x_i of \mathfrak{m} . Show that $I\cdot k[U_i]$ can be written as $I_{E\cap U_i}^d\cdot I_{\widetilde{X}\cap U_i}$ where d is the degree of f and the ideals I_{\bullet} are taken inside $k[U_i]$.

(b) In the setting of (a), show that $U_i \cap E$ are identified with the standard affine charts on $E \cong \mathbb{P}^n$. Further show that $Z \cong \widetilde{X} \cap E$ and that $\widetilde{X} \cap E \cap U_i$ correspond to the standard affine charts on Z.

(c) Show in general that $Z \cong \widetilde{X} \cap E$.

¹Locus on \widetilde{Y} where π is not an isomorphism