WORKSHEET #6 – MATH 6130
FALL 2018

DUE WEDNESDAY, OCTOBER 31ST

You may work in groups of up to 3. Only one worksheet needs to be turned in per group.

1. Suppose that X is an affine variety with $R = k[X]$. Consider an ideal $I \subseteq R$. Show that the blowup of I is isomorphic to the blowup of I^2.

Hint: The blowup lives in $X \times \mathbb{P}^n$ where $I = (f_0, \dots, f_n)$. I^2 is generated by $n^2 + 2n + 1$ elements $(f_0^2, f_0 \cdot f_1, \dots, f_n^2)$. Note on the other hand we have the Segre embedding $\mathbb{P}^n \longrightarrow \mathbb{P}^n \times \mathbb{P}^n \subseteq P^N$ where $N = n^2 + 2n$.

2. Consider $X = \mathbb{A}^2$ with $k[X] = k[x, y]$. Show that there is a map from the blowup of (x^2, xy, y^2) (which is isomorphic to the blowup of (x, y) by **1.**), to the blowup of (x^2, y^2). Is it an isomorphism?

3. Suppose that X is an affine variety and $I, J \subseteq R = k[X]$ are ideals. Prove that there is a map from Z, the blowup of $I \cdot J$, to Y, the blowup of I. $\nu : Z \longrightarrow Y$.

4. With notation as in **3.**, after picking generators f_i for I with corresponding affine charts $U_i \subseteq Y$, show $\nu^{-1}(U_i) \longrightarrow U_i$ is the blowup of the extended ideal $J \cdot k[U_i]$ (extended via the map $k[X] \longrightarrow k[U_i]$).

5. Consider $X = Z(x \cdot y - z^2) \subseteq \mathbb{A}^3$. Consider $\pi : Y \longrightarrow X$ the blowup of the ideal $I = (x, z)$. Find the locus on which π is an isomorphism and show it is strictly larger than $X \setminus Z_X(I)$.

6. Suppose that $X \subseteq Y$ is a closed subvariety of an affine variety. Consider an ideal $I \subseteq k[x_1, \ldots, x_n] = k[\mathbb{A}^n]$ and let $\pi : \widetilde{Y} \longrightarrow Y$ be the blowup of I. Let $\widetilde{X}$ be the strict transform of X in Y. Prove (without looking *too* closely at the book, which also proves this) that $\pi|_{\widetilde{X}} : \widetilde{X} \longrightarrow X$ is the blowup of $I \cdot k[X]$.

7. Suppose that I is a homogenous prime ideal in $k[x_0, \dots, x_n] = R$ not containing $\mathfrak{m} = (x_0, \dots, x_n)$. Consider the associated variety $X \subseteq \mathbb{A}^{n+1}$ (this is called the affine cone over the projective variety $Z = Z_{\mathbb{P}^n}(I) \subseteq \mathbb{P}^n$ – try drawing a picture, because I obviously can't). Consider the blowup of $\mathfrak{m}$, $\pi : \widetilde{Y} \longrightarrow \mathbb{A}^{n+1}$ and let $E \cong \mathbb{P}^n = \pi^{-1}(\text{origin})$ be the exceptional set[1]. Further let $\widetilde{X} \subseteq \widetilde{Y}$ be the strict transform of $tldX$.

(a) If $I = (f)$ is a principal ideal, consider the affine charts $U_0, \dots, U_n \subseteq \widetilde{Y}$ corresponding the generators x_i of $\mathfrak{m}$. Show that $I \cdot k[U_i]$ can be written as $I^d_{E \cap U_i} \cdot I_{\widetilde{X} \cap U_i}$ where d is the degree of f and the ideals $I_\bullet$ are taken inside $k[U_i]$.

(b) In the setting of (a), show that $U_i \cap E$ are identified with the standard affine charts on $E \cong \mathbb{P}^n$. Further show that $Z \cong \widetilde{X} \cap E$ and that $\widetilde{X} \cap E \cap U_i$ correspond to the standard affine charts on Z.

(c) Show in general that $Z \cong \widetilde{X} \cap E$.

[1]Locus on $\widetilde{Y}$ where π is not an isomorphism