WORKSHEET #4 – MATH 6130
FALL 2018

DUE FRIDAY, SEPTEMBER 28TH

You may work in groups of up to 3. Only one worksheet needs to be turned in per group.

1. Suppose $q = (a_0 : \ldots : a_n) \in \mathbb{P}^n_k$ is a point and $k[x_0, \ldots, x_n] = S(\mathbb{P}^n_k) =: T$. Write down a set of generators for the ideal $I(q) \subseteq T$. Is it a maximal or prime ideal? Is it maximal with respect to some condition? If so, what?

2. Let $I = (F_1 = y_1y_2 - y_3, F_2 = y_1^2 - y_2) \subseteq k[y_1, y_2, y_3]$. Show that I is a prime ideal and show that (F_1^h, F_2^h) is not equal to I^h.

Hint: Recall the ideal of the twisted cubic $J = (x_1x_2 - x_0x_3, x_1^2 - x_0x_2, x_2^2 - x_1x_3)$.

Let's do a commutative algebra problem or two.

3. Suppose that $A = \oplus_{i\geq 0} A_i$ is a graded ring where $A_0 = K$ is a field. If A is a finitely generated $A_0 = K$-algebra, show that there exists an integer n such that $A^{(n)} = \bigoplus_{j\geq 0} A_{nj}$ is a standard graded K-algebra (in other words, generated in degree 1).

4. Suppose that $A = \oplus_{i\geq 0} A_i$ is a graded ring. Consider the subring $A^{(n)} \subseteq A$ as in the previous problem. Suppose that $x \in A$ is a homogeneous element. Show that

$$A^{(n)}_{(x^n)} \cong A_{(x)}.$$

Rcall the following.

Definition. A regular map $\phi : Y \longrightarrow X$ between quasi-projective varieties is a continuous map such that for every open set $U \subseteq Y$ the map $\phi^* : \mathrm{Map}(U, k) \longrightarrow \mathrm{Map}(\phi^{-1}(U), k)$ induces a k-algebra homomorphism $\mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(\phi^{-1}(U))$.

The following proposition from the book (3.39) is also helpful as it lets us check locally that a map is regular.

Proposition. *Suppose that X and Y are quasi-projective varieties and $\phi : X \longrightarrow Y$ is a map. Let $\{V_i\}$ be a collection of open affine subsets covering Y and $\{U_{ij}\}$ a collection of open subseteq covering X, such that*

(1) $\phi(U_{ij}) \subseteq V_i$ for all i, j and

(2) the map $\phi^* : \mathrm{Map}(V_i, k) \longrightarrow \mathrm{Map}(U_{ij}, k)$ induces a k-algebra homomorphism $\mathcal{O}_Y(V_i) \longrightarrow \mathcal{O}_X(U_{ij})$ for all i, j.

Then ϕ is a regular map.

Recall that a rational map between quasi-affine varieties X and Y is a map only defined on an open subset of X. We make the same definition with projective varieties.

5. Suppose that $h_0, \ldots, h_m \subseteq k[x_0, \ldots, x_n]$ are homogeneous elements of the same degree $d > 0$. Show that

$$\begin{array}{ccc} \mathbb{P}^n & \dashrightarrow & \mathbb{P}^m \\ (a_0 : \ldots : a_n) & \longmapsto & (h_0(a_0, \ldots, a_n) : \ldots : h_m(a_0, \ldots, a_n)) \end{array}$$

is a rational map. You will need to local an open set on which is is (well) defined and verify it really is regular on that set.

Hint: If you want to use the proposition, you can start by looking at some open sets $U_j = D(x_j)$. But they might not map into $D(y_j)$. But you can intersect them with the pre-images of $D(y_j)$.

You might need an extra page to write down the details.

6. With the notation of the previous problem, suppose that $\sqrt{(h_0, \ldots, h_m)} = (x_0, \ldots, x_m)$ (equality here is as ideals). Show that the induced map is defined everywhere.

7. Consider the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by $(x : y : z) \mapsto (yz : xz : xy)$. Find where this map is not defined.