WORKSHEET #3 – MATH 6130
FALL 2018

DUE MONDAY, SEPTEMBER 17TH

You may work in groups of up to 3. Only one worksheet needs to be turned in per group.

1. Consider $U = \mathbb{A}^2 \setminus \{0\}$. Compute $R = \mathcal{O}_{\mathbb{A}^2}(U)$ as a subring of $k(x,y)$, the fraction field of $\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2) = k[x,y]$.

Hint: There are different ways to do this. One way is to use the fact that $k[x,y]$ is a UFD and suppose that $f/g \in R$, with $f, g \in k[x,y]$ having no common factors. You should show that this expression has to be essentially unique. On the other hand, we know $\dim Z(g) = 1$.

2. Prove that $U = \mathbb{A}^2 \setminus \{0\}$ is not isomorphic to an affine variety.

3. Consider the regular map $\phi : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$ defined by $\phi(x,y) = (x, xy)$. Show that ϕ is dominant, birational, not an isomorphism and not finite.

4. Suppose that $X \subseteq \mathbb{A}^n$ is a non-empty closed irreducible set such that $I(X) = (f_1, \ldots, f_r)$ is generated by r elements. Show that $\operatorname{codim}_{\mathbb{A}^n}(X) \leq r$.

Hint: Remember that $\operatorname{codim}_{\mathbb{A}^n}(X) := \dim \mathbb{A}^n - \dim X$. We also may use that if $f \in k[Y]$ is a nonzero divisor on some algebraic variety $Y \subseteq \mathbb{A}^n$, then $\dim Z_Y(f) = \dim Y - 1$ (Theorem 2.7.1 in the text). The issue that makes this slightly tricky is that $Z_Y(f)$ is not necessarily itself an algebraic variety (there is no reason it must be irreducible).