DIMENSION THEORY

Recall the following notions of dimension.

(a) For a Noetherian local ring $(R, \mathfrak{m})$, we define $\chi_R(n) = \lambda(R/\mathfrak{m}^n)$ and recall it is eventually a polynomial. Then we set $d(R)$ to be the (eventual) degree of $\chi_R(n)$.
(b) For a Noetherian local ring $(R, \mathfrak{m})$, we define $\delta(R) = \min\{\# \text{ of generators of } \mathfrak{q} \mid \sqrt{\mathfrak{q}} = \mathfrak{m}\}$.
(c) For any ring R, we define the *Krull dimension of* R to be maximum length n of a chain of prime ideals $P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n$ (or we define it to be infinity if no maximal length chain exists)

On Monday, we showed that these three notions of dimension were equivalent.

1. Suppose that $(R, \mathfrak{m})$ is a Noetherian local ring, prove that $\dim(R)$ is finite.

Aside: It turns out that there are Noetherian non-local rings that have infinite Krull dimension (although you wouldn't want to meet any in a dark alley at night).

2. Suppose that R is a Noetherian ring and $x_1, \ldots, x_r \in R$. Suppose that $\mathfrak{q}$ is a minimal associated prime of R/I (ie, the radical of a minimal primary ideal in a primary decomposition of I). Show that $\mathfrak{q}$ has height $\leq r$ (recall that the height of $\mathfrak{q}$ is the dimension of $R_{\mathfrak{q}}$).

Hint: Consider $\delta(R_{\mathfrak{q}})$.

3. Prove that $k[x_1, \ldots, x_n]$ has dimension n.

Hint: You can reduce to the case where k is algebraically closed since $k \subseteq \overline{k}$ is integral. Then use **2.** to show that $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$ has height n (it clearly has height at least n from a chain of primes as we discussed previously).

4. Suppose that R is a Noetherian ring and $x \in R$ is not a zero divisor or a unit. Show that every minimal associated prime of $R/\langle x \rangle$ has height 1.

5. Suppose that R is a Noetherian local ring and x is an element of $\mathfrak{m}$ which is a regular element of R. Show that $\dim(R/\langle x \rangle) = \dim R - 1$.

Hint: We already know the inequality $\leq$ from class (for $d(R)$). Use the work above for the other inequality.

6. Suppose that k is a field and R is a domain of finite type over a field. If $x_1, \ldots, x_d$ is a transcendence basis for the fraction field R over k, prove that $\dim R = d$. Conclude that every maximal ideal of R has the same height.

Hint: Use Noether normalization.

7. Consider the ring $R = k[x, y, z]/(\langle x \rangle \cap \langle y, z \rangle)$. Determine the dimension of R. Find two different maximal ideals of different heights. What is the dimension of $R/\langle x \rangle$? What is the dimension of $R/\langle y \rangle$?