HOMEWORK #7 – MATH 538
FALL 2013

DUE MONDAY, DECEMBER 9TH

(1) Suppose that $(R, \mathfrak{m})$ is a regular local ring with $\dim R = d$ and $\mathfrak{m} = \langle f_1, \dots, f_d \rangle$. Show that $R/\langle f_1, \dots, f_i \rangle$ is also regular for each i.

(2) Suppose that $(R, \mathfrak{m})$ is a local ring and $f \in R$ is a non-zero divisor and non-unit such that $R/\langle f \rangle$ is regular. Prove that R is regular.

(3) Suppose that $S = \mathbb{C}[x_1, \dots, x_n]$ and $f \in \mathfrak{m} = \langle x_1, \dots, x_n \rangle \in S$. Suppose that $V(f)$ is a manifold at the origin (ie, $\frac{\partial f}{\partial x_i}(\mathbf{0}) \neq 0$ for some i). Show that $R = S_{\mathfrak{m}}/\langle f \rangle$ is a regular local ring. Further suppose that $\mathfrak{n}$ is the maximal ideal of R. Show that $\mathfrak{n}/\mathfrak{n}^2$ can be canonically identified with the $\mathbb{C}$-dual of the tangent space of $V(f)$ at the origin.

Aside: You may also want to read Theorem 5.1 in Chapter I of Hartshorne, for some additional discussion.

(4) Suppose that $R = R_0 \oplus R_1 \oplus R_2 \oplus R_3 \dots$ is a graded Noetherian ring and set $R_+ = 0 \oplus R_1 \oplus R_2 \oplus \dots$, the *irrelevant ideal of* R.

  (a) An ideal $I \subseteq R$ is called *homogeneous* if I can be generated by homogeneous elements (ie, if $I = \langle f_1, \dots, f_m \rangle$ where each $f_j \in R_{d_j}$ lives in a single degree). Show that I is homogeneous if and only if $I = I_0 \oplus I_1 \oplus I_2 \oplus \dots$ where $I_i = I \cap R_i$.

  (b) We define $\operatorname{Proj} R$ to be the set of homogeneous prime ideal that do not contain R_+, in other words

$$\operatorname{Proj} R = \{P \in \operatorname{Spec} R \mid R_+ \not\subset P, P \text{ is homogeneous}\}.$$

  For any homogeneous ideal $I \subseteq R$ we define

$$V(I) = \{P \in \operatorname{Proj} R \mid I \subseteq P\}$$

  Show that the sets $V(I)$ define the closed sets of a topology on $\operatorname{Proj} R$.

  (c) Prove that the maximal homogeneous ideals of $\mathbb{C}[x_0, \dots, x_n]$ (with the usual grading, each x_i has degree 1) are in bijection with the points of $\mathbb{P}^n(\mathbb{C})$ (projective n-space over $\mathbb{C}$).

  (d) For each homogeneous element $f \in R_i$, we define[1] $R_{(f)}$ to be the subring of $R_f = \{1, f, f^2, \dots\}^{-1} R$ of elements of degree exactly zero (for instance, if $g, f \in R_2$, then $g/f \in R_{(f)}$). Show that there is a bijection between $\operatorname{Spec} R_{(f)}$ and $(\operatorname{Proj} R) \setminus V(f)$.

  Aside: It turns out that the $\operatorname{Spec} R_{(f)}$ form an affine cover for the scheme $\operatorname{Proj} R$.

(5) Suppose that $S = k[x, y]$ and consider the Rees algebra (a graded ring)

$$R = k[x, y] \oplus \langle x, y \rangle \oplus \langle x, y \rangle^2 \oplus \langle x, y \rangle^3 \oplus \dots$$

Describe $\operatorname{Proj} R$ geometrically.

[1]Yes, the notation is horrible!