HOMEWORK #2 – MATH 538
FALL 2013

DUE FRIDAY, SEPTEMBER 20TH

(1) Suppose that $W_1 \subseteq W_2$ are multiplicative sets in a ring R and M is an R-module. Show that $W_2^{-1}(W_1^{-1}M) \cong W_2^{-1}M$.

(2) Suppose that $N \subseteq M$ are R-modules and $W \subseteq R$ is a multiplicative set. Show that $W^{-1}(M/N) \cong W^{-1}M/W^{-1}N$. Here we are using the fact that $W^{-1}N$ can be identified with a submodule of $W^{-1}M$.

(3) Suppose that R and S are integral domains and $f : R \longrightarrow S$ is a ring homomorphism. Show that f is injective if and only if the induced map $\rho : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ has dense image (in the topology of $\operatorname{Spec} R$). In general, if a map $\phi : Y \longrightarrow X$ of topological spaces has dense image in its codomain, then ϕ is called *dominant*.
  (i) Give an example of non-domains where $f : R \longrightarrow S$ is not injective but $\rho : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is dominant.
  (ii) Weaken the hypotheses that R and S are domains as much as you can while still keeping the equivalence of the injectivity of f and the dominance of ρ. (There isn't a single right answer, although there is probably a best answer).

(4) We consider localizing at prime ideals.
  (a) Suppose that A is a ring and that for each prime ideal $P \in \operatorname{Spec} A$, the local ring $A_P := (A \setminus P)^{-1}A$ is an integral domain. Show that A need not be an integral domain.
  (b) However, if M and N are A-modules with a map $f : M \longrightarrow N$. Consider the induced map $f_P : (A \setminus P)^{-1}M =: M_P \longrightarrow N_P := (A \setminus P)^{-1}N$ for each $P \in \operatorname{Spec} A$. Show that f is surjective (respectively injective) if and only if f_P is surjective (respectively injective) for every $P \in \operatorname{Spec} A$.

(5) Suppose R is a ring and I is an ideal. Show that the set $\sqrt{I} := \{r \in R \mid r^n \in I \text{ for some } n\}$ is an ideal, it is called the *radical of* I. In the case that $I = \langle 0 \rangle$, $\sqrt{0}$ is called the nilradical.

(6) Let A be a ring and $\mathfrak{n} = \sqrt{0}$ is its nilradical. Show that the following are equivalent:
  (i) A has exactly one prime ideal.
  (ii) Every element of A is either a unit or nilpotent.
  (iii) $A/\mathfrak{n}$ is a field.

(7) Suppose that k is a field and that $f : R \longrightarrow S$ is a map between finitely generated k-algebras (this means that R is of the form $k[x_1, \dots, x_n]/I$, and likewise with S, and also that f sends k to k). Show that the function $f^{\#} : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ sends maximal ideals to maximal ideals. *Hint:* You may use the fact that if K is a field and $L \supseteq K$ a field extension such that L is a finitely generated K-algebra, then L is a finite field extension. You may also use that an integral domain which is a finite extension of a field is itself a field.

(8) Suppose that A and B are rings. Figure out what $\operatorname{Spec}(A \oplus B)$ is.

(9) Suppose that A and B are rings, I is an ideal of A with canonical projection $f : A \longrightarrow A/I$. Further suppose that we are given a ring homomorphism $g : B \longrightarrow A/I$. Consider the following set.

$$C = \{(a, b) \in A \oplus B | f(a) = g(b)\}.$$

  (a) Show that C is a subring of $A \oplus B$. Note that C has canoical maps to A and to B (call them p_1 and p_2 respectively).

(b) Show that the elements of $\operatorname{Spec} C$ are in bijection with the set

$$\Big((\operatorname{Spec} A) \setminus V(I)\Big) \coprod (\operatorname{Spec} B).$$

Hint: Consider a prime in $\operatorname{Spec} C$. There are two possibilities, it either contains $\ker p_2$ or it does not. In the latter case, invert an appropriate element and analyze what happens.

(c) Describe geometrically $\operatorname{Spec} C$ in the following examples where k is an algebraically closed field. Additionally describe the map $\operatorname{Spec} A \longrightarrow \operatorname{Spec} C$ in each case:

(i) $A := k[x]$, $I = \langle x^2 - 1 \rangle$ and $B = k$.

(ii) $A := k[x, y]$, $I = \langle x \rangle$ and $B = k$.

(d) Suppose now that g is surjective. Explain why $\operatorname{Spec} C$ is the gluing of $\operatorname{Spec} A$ to $\operatorname{Spec} B$ along $V(I)$.

The following philosophical statement on the elements of C might help with this problem. C is made up of the functions in $A \oplus B$ that agree on the set $V(I)$.