WORKSHEET # 4 SOLUTIONS

MATH 538 FALL 2011

Our goal in this worksheet is to play with faithful flatness. Recall the following definition:

Definition 0.1. If R is a ring and M is an R-module, we say that M is *faithfully flat* if M is flat and if for any two R-modules N and N', the natural map

$$\mathrm{Hom}_R(N, N') \to \mathrm{Hom}_R(N \otimes M, N' \otimes M)$$

is injective. Given $\phi \in \mathrm{Hom}_R(N, N')$, we will use $\widetilde{\phi}$ to denote the corresponding element of $\mathrm{Hom}_R(N \otimes M, N' \otimes M)$.

First however we do a warm-up problem on flatness.

1. Suppose that S is a flat R-algebra. Then $\mathrm{Hom}_R(N', N) \otimes S \cong \mathrm{Hom}_S(N' \otimes S, N \otimes S)$ for any R-modules N, N'. FURTHERMORE, assume that R is Noetherian and N' is finitely generated OR simply that N' is finitely presented.

Hint: Verify the theorem if N' is a finite free-module. Then notice that for any module N' there exists an exact sequence: $F_1 \to F_0 \to N' \to 0$ with F_i free modules and recall that $\mathrm{Hom}_R(\bullet, N)$ will turn this into another exact sequence (this time exact on the left).

Solution: Assume first that $N' = \oplus R^n = R^{\oplus n}$. Then $\mathrm{Hom}_R(R^n, N) \otimes_R S = N^n \otimes S = (N \otimes S)^n = \mathrm{Hom}_S(S^n, N \otimes S)$ (this isomorphism is also the obvious one if one fixes a basis for R^n).

Now in the general case, choose an exact sequence

$$R^m \xrightarrow{f} R^n \xrightarrow{g} N' \to 0.$$

Note that being finitely presented exactly means that such a sequence exists (with m and n finite), and its easy to any finitely generated module is finitely presented for R-Noetherian.

We now also observe that there is a natural map $\mathrm{Hom}_R(N', N) \otimes_R S \to \mathrm{Hom}_S(N' \otimes S, N \otimes S)$ where $\phi \otimes a$ is sent to $a.\widetilde{\phi}$. The fact that it is natural here means that there is a map $\mathrm{Hom}_R(\bullet, N) \otimes_R S \to \mathrm{Hom}_S(\bullet \otimes S, N \otimes S)$ which respects compositions. I won't check this.

Now we apply the functors $\mathrm{Hom}_R(\bullet, N) \otimes_R S$ and $\mathrm{Hom}_S(\bullet \otimes S, N \otimes S)$ to this exact sequence and we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(N', N) \otimes S \arrow[r] \arrow[d] & \mathrm{Hom}_R(R^n, N) \otimes S \arrow[r] \arrow[d, "\sim"'] & \mathrm{Hom}_R(R^m, N) \otimes S \arrow[d, "\sim"'] \\
0 \arrow[r] & \mathrm{Hom}_S(N' \otimes S, N \otimes S) \arrow[r] & \mathrm{Hom}_S(R^n \otimes S, N \otimes S) \arrow[r] & \mathrm{Hom}_S(R^m \otimes S, N \otimes S)
\end{tikzcd}$$

The two right vertical arrows are surjective by our earlier work. It follows that $\mathrm{Hom}_R(N', N) \otimes S \cong \mathrm{Hom}_S(N' \otimes S, N \otimes S)$ since they are both kernels of the same map (up to isomorphism).

2. Suppose that $M = S$ is a FLAT R-algebra. Prove that S is faithfully flat if and only if for any N, N', the natural map

$$\mathrm{Hom}_R(N, N') \to \mathrm{Hom}_S(N \otimes_R S, N' \otimes_R S)$$

is injective. This should be very easy (notice the second Hom is now over S).

Solution: We have the factorization:

$$\mathrm{Hom}_R(N, N') \to \mathrm{Hom}_S(N \otimes_R S, N' \otimes_R S) \hookrightarrow \mathrm{Hom}_R(N \otimes_R S, N' \otimes_R S)$$

S being faithfully flat means that the composition is injective. This certainly implies that the stated map is injective. Conversely, if the stated map is injective, the map used to determine faithful flatness is also injective.

We now move on to our main goal.

Theorem 0.2. *Fix R to be a ring, S to be a FLAT R-algebra and M, N', N, N'' to be other R-modules. The following are equivalent (TFAE):*

(i) *S is a faithfully flat.*
(ii) *$N' \xrightarrow{f} N \xrightarrow{g} N''$ is exact if and only if $N' \otimes S \xrightarrow{f_S} N \otimes S \xrightarrow{g_S} N'' \otimes S$ is exact.*
(iii) *For any ideal $\mathfrak{a} \subseteq R$, $(\mathfrak{a} \cdot S) \cap R = \mathfrak{a}$.*
(iv) *$\operatorname{Spec} S \to \operatorname{Spec} R$ is surjective.*
(v) *For every maximal ideal $\mathfrak{m} \subseteq R$, we have $\mathfrak{m} \cdot S \neq S$.*
(vi) *If $N \neq 0$, then $N \otimes S \neq 0$.*
(vii) *The map $M \to M \otimes S$ (defined by $x \mapsto x \otimes 1$) is injective.*

3. Prove that (iii) implies (iv) by first showing that if $A \hookrightarrow B$ is a ring homomorphism and $\mathfrak{p}$ is a prime ideal of A, then there exists a prime ideal $\mathfrak{q}$ of B such that $\mathfrak{q} \cap A = \mathfrak{p}$ as long as $(\mathfrak{p} \cdot B) \cap A = \mathfrak{p}$. (the converse to this theorem holds too, but we don't need it)

Solution: We first prove the claim. So suppose that $\mathfrak{p} \in \operatorname{Spec} A$ and that $(\mathfrak{p} \cdot B) \cap A = \mathfrak{p}$. Set $W = A \setminus \mathfrak{p}$. It follows that $(\mathfrak{p} \cdot W^{-1}B) \cap (W^{-1}A) = \mathfrak{p} \cdot (W^{-1}A)$. Choose any prime ideal of $W^{-1}B$ containing $\mathfrak{p}$ (note one must exist) and call it $\mathfrak{q}'$. Note that $\mathfrak{q}' \cap (W^{-1}A)$ is a prime ideal of $W^{-1}A$ which contains the maximal ideal of $W^{-1}A$. Thus $\mathfrak{q}' \cap (W^{-1}A) = \mathfrak{p}$. We know $\mathfrak{q}'$ corresponds to a prime ideal $\mathfrak{q} \in \operatorname{Spec} B$ and so we also have that $\mathfrak{q} \cap A = \mathfrak{p}$ and the claim is proven.

Now we move on to proving that (iii) implies (iv). But this is easy, for any $\mathfrak{p} \in \operatorname{Spec} R$, choose $\mathfrak{a} = \mathfrak{p}$ and apply (iii) and the claim.

4. Show that (iv) easily implies (v). Then show that (v) implies (vi) as follows: choose $x \in N$ and set $N' = \langle x \rangle$. It is enough to show that $N' \otimes S \neq 0$ (why?). Note $N' = R/\mathfrak{a}$, $\mathfrak{a}$ is contained in some maximal ideal...

Solution: For (iv) implies (v), if $\mathfrak{m} \in R$ is maximal, then there exists $\mathfrak{q} \in \operatorname{Spec} S$ such that $\mathfrak{q} \cap R = \mathfrak{m}$. In particular $\mathfrak{m} \subseteq \mathfrak{m} \cdot S \subseteq \mathfrak{q} \subsetneq S$.

For (v) implies (vi), we follow the hint: Note that since S is flat, $N' \otimes S \subseteq N \otimes S$ and so if the left side is non-zero, so is the right. Choose $\mathfrak{n} \in \operatorname{Spec} R$ a maximal ideal containing $\mathfrak{a}$. Now suppose that $(R/\mathfrak{a}) \otimes S = 0$. Then since $(R/\mathfrak{a})$ surjects onto $(R/\mathfrak{n})$ and $\bullet \otimes_R S$ is right exact, we also have that $0 = (R/\mathfrak{a}) \otimes S \twoheadrightarrow (R/\mathfrak{n}) \otimes S = (S/\mathfrak{n}S)$. Thus $S = \mathfrak{n}S$ and so (v) is false.

Assume without proof that (vi) implies (vii).

5. Now prove that (vii) implies (iii) by choosing $M = R/\mathfrak{a}$. This proves that (iii) to (vii) are equivalent.

Solution: Choosing $\mathfrak{a} \subseteq R$ and $M = R/\mathfrak{a}$, suppose that $M \to M \otimes S$ is injective. Thus $R/\mathfrak{a} \to (R/\mathfrak{a}) \otimes S = S/\mathfrak{a}S$ is injective. But the kernel of that map is exactly $\big((\mathfrak{a} \cdot S) \cap R\big)/\mathfrak{a}$, and so proving that the kernel is zero proves the desired equality.

6. Prove that (ii) and (vi) are equivalent.

Hint: If (vi) is false, concoct an exact sequence. If (vi) is true, suppose that $N' \otimes S \xrightarrow{f_S} N \otimes S \xrightarrow{g_S} N'' \otimes S$ is exact. So $\operatorname{Im}(g \circ f) \otimes S = \operatorname{Im}(g_S \circ f_S) = 0$. Thus we get $\ker f \supseteq \operatorname{Im} g$. Now consider $\ker f / \operatorname{Im} g$ and tensor with S.

Solution: Suppose that (vi) is false with N being the module in question. Then $0 \to N \to 0$ is not exact but $0 \otimes S \to N \otimes S \to 0 \otimes S$ is exact.

The ($\Rightarrow$) direction of (ii) follows from flatness and so we get that for free. Conversely, suppose that (vi) is true and follow the hint. We get $\ker f \subseteq \operatorname{Im} g$ so that's half the battle for exactness. If the map is not exact, then $\ker f / \operatorname{Im} g \neq 0$. Then tensoring with S we obtain that

$$0 \neq (\ker f / \operatorname{Im} g) \otimes S \cong \ker f_S / \operatorname{Im} g_S$$

(the isomorphism comes from the right exactness of tensor). But $\ker f_S = \operatorname{Im} g_S$ by hypothesis and so this is a contradiction.

7. Now we start to tackle (i). Assume (vi) does not hold for N. Prove that (i) does not hold by considering $\mathrm{Hom}_R(R, N)$.

Solution: Consider the natural map $\mathrm{Hom}_R(R, N) \to \mathrm{Hom}_S(R \otimes S, N \otimes S)$. We will show that this map is not injective. Now $0 \neq N \cong \mathrm{Hom}_R(R, N)$ and $0 = N \otimes S$ so that $0 = \mathrm{Hom}_S(R \otimes S, N \otimes S) = \mathrm{Hom}_S(S, 0)$ and so the map cannot be injective.

8. Prove that (vii) implies (i) The original hint was bogus.

Solution: Consider the composition $M = \mathrm{Hom}_R(N, N') \to M \otimes S = \mathrm{Hom}_R(N, N') \otimes S \to \mathrm{Hom}_R(N \otimes S, N' \otimes S)$. The first map is injective by (vi) but this is not enough, N need not be finitely presented so we can't use problem **1.**

So fix $\phi \in M = \mathrm{Hom}_R(N, N')$. We have an exact sequence:

$$0 \to K \to N \xrightarrow{\phi} N' \to C \to 0$$

Tensoring with S we obtain:

$$0 \to K \otimes S \to N \otimes S \xrightarrow{\widetilde{\phi}} N' \otimes S \to C \otimes S \to 0$$

Since S is flat, this is also exact. Now, if $\widetilde{\phi}$ is the zero map, then $K \otimes S \to N \otimes S \to 0$ would be exact and so $K \to N \to 0$ would be exact and so ϕ would necessarily be the zero map.