WORKSHEET # 3

MATH 538 FALL 2011

In this worksheet, we'll explore valuation rings. The content of this worksheet is taken largely from Atiyah-Macdonald.

Definition 0.1. Suppose that B is an integral domain and $K = K(B)$ is its field of fractions. We say that B is a *valuation ring of* K if for each $0 \neq x \in K$, either we have $x \in B$ or $x^{-1} \in B$.

1. Show that any valuation ring is local.

Solution: It is sufficient to show that the set of non-units of b is an ideal. Let us use $\mathfrak{m}$ to denote this set.

Suppose first that $x \in \mathfrak{m}$ and $b \in B$. If $bx \notin \mathfrak{m}$, then bx is a unit and so x is a unit as well, a contradiction.

Suppose that $x, y \in \mathfrak{m}$ consider $x + y$. Either $x/y \in B$ or $y/x \in B$. Without loss of generality suppose the former. Then $\mathfrak{m}$ contains $y((x/y) + 1) = x + y$ as desired.

2. Show that if B is a valuation ring and A is any other ring such that $B \subseteq A \subseteq K$, then A is also a valuation ring.

Solution: This is obvious.

3. Show that any valuation ring is normal (in other words, it's integral closure in K is itself).

Solution: Suppose that $x \in K$ is integral over B. Thus there exist $b_{n-1}, \ldots, b_0$ such that

$$x^n + b_{n-1}x^{n-1} + \cdots + b_1 x^1 + b_0 = 0.$$

If $x \notin B$, then $x^{-1} \in B$ and multiplying through by $x^{-(n-1)}$ yields

$$x^1 = -(b_{n-1} + b_{n-2}x^{-1} + \cdots + b_1 x^{-(n-2)} + b_0 x^{-(n-1)} \in B$$

a contradiction.

4. Show that any 1-dimensional local domain R with maximal ideal $\mathfrak{m} = \langle x \rangle$, is a valuation ring.
Hint: You may assume without proof that a Noetherian 1-dimensional local domain is normal if and only if it is a UFD if and only if $\mathfrak{m} = \langle x \rangle$. Also recall that all elements in such rings are of the form ux^n for u a unit.

Solution: Indeed, all elements of the ring are of the form ux^n, $n \geq 0$ and u a unit of R. Thus every element of $K = K(R)$ is of the form ux^n for $n \in \mathbb{Z}$ and u a unit of R. The result follows.

5. Suppose that B is a valuation ring with $K = K(B)$. Let U denote the group of units of B, it is a subgroup of the multiplicative group $K^* = K \setminus \{0\}$. Set $\Gamma = K^*/U$.

Given $[x], [y] \in \Gamma$ define $[x] \geq [y]$ if $xy^{-1} \in B$. Show that this is a total ordering of Γ such that if $[x] \geq [y]$ then $[x][z] \geq [y][z]$ (in other words, Γ is a totally ordered Abelian group). Let $v : K^* \to \Gamma$ denote the natural surjection. Show that $v(x + y) \geq \min(v(x), v(y))$.

The group Γ is called a *value group for B*.

Solution: First suppose that $[x] \sim [x']$. Thus $x = ux'$ for some unit $u \in B$ and so $xy^{-1} \in B$ if and only if $x'y^{-1} \in B$. The same argument holds if $[y] \sim [y']$ and so the relation $\geq$ is well defined. If $[x] \geq [y]$ and $[y] \geq [z]$ then $xy^{-1} \in B$ and $yz^{-1} \in B$ so $xz^{-1} = xy^{-1}yz^{-1} \in B$ and the relation $\geq$ is transitive. The fact that any two items are comparable follows immediately from the definition of a valuation ring. Thus we do indeed have a total ordering.

Now suppose that $[x] \geq [y]$. Then $xy^{-1} \in B$ so that $xz(yz)^{-1} \in B$ and so $[x][z] = [xz] \geq [yz] = [y][z]$.

Now, we also observe that $v(1)$ is the identity element of the group Γ and that $v(1) = v(y)$ if and only if y is a unit of B. Further note that if $v(x) \geq v(1)$, then $x \in B$. Finally, consider $v(x + y) = [x + y]$. Note that either x/y or $y/x \in B$. Suppose the former so that $v((x/y) + 1) \geq v(1)$. Then $x + y = y((x/y) + 1)$ so $v(x + y) = v(y)v((x/y) + 1) \geq v(y)v(1) = v(y)$. Alternately, if $y/x \in B$ then $v(1 + (y/x)) \geq v(1)$ and then $v(x + y) = v(x)v(1 + (y/x)) \geq v(x)$

6. Show that for a ring as in **4.**, Γ as in **5.** is an infinite cyclic group ordered like the integers. Because of this such rings are called *discrete valuation rings*.

Solution: Indeed, if $\mathfrak{m} = \langle x \rangle$, then since every element of $K = K(R)$ is of the form ux^n for $u \in R$ a unit and $n \in \mathbb{Z}$, we see that $\Gamma = \{\dots, [x^{-2}], [x^{-1}], [1], [x^1], [x^2], \dots\}$. The ordering is obvious since $[x^{n+1}] \geq [x^n]$.

We now prove something like a converse to **5.** which also justifies our original terminology (valuation ring).

7. Suppose that Γ is an arbitrary totally ordered Abelian group (written multiplicatively) and that $v : K^* \to \Gamma$ is a function such that

(i) $v(xy) = v(x)v(y)$, and

(ii) $v(x + y) \geq \min(v(x), v(y))$

for all $x, y \in K^*$. Show that the set

$$R := \{x \in K^* \mid v(x) \geq v(1)\} \cup \{0\}$$

is a valuation ring.

Solution: First we note that (ii) certainly implies that R is closed under addition. Also, if $x, y \in R$, then $v(xy) = v(x)v(y) \geq v(x)v(1) = v(x) \geq v(1)$ and so $xy \in R$ and so R is closed under multiplication. It certainly has multiplicative and additive identities (after I fixed the typo above). Thus it is a ring.

Now, for any $x \in K^*$, if $x \notin R$ so that $v(x) \ngeq v(1)$ then $v(1) = v(x)v(x^{-1}) \geq v(1)v(x^{-1}) = v(x^{-1})$ and so $x^{-1} \in R$ as desired.