WORKSHEET # 2

MATH 538 FALL 2011

In this worksheet, we'll explore normalization of an integral domain.

Definition 0.1. Suppose R is an integral domain with field of fractions $K(R)$. We define the *normalization of* R to be the integral closure of R inside $K(R)$. We will denote this by R^{N}. We say that R *is normal* if $R = R^{\mathrm{N}}$.

1. Show that $K(R) = K(R^{\mathrm{N}})$.
Hint: This should be essentially trivial.

Solution: Indeed, certainly $R^{\mathrm{N}} \subseteq K(R)$. Thus we have the containments

$$K(R) = \mathrm{Frac}(R) \subseteq \mathrm{Frac}(R^{\mathrm{N}}) \subseteq K(R)$$

proving the result

2. Suppose that R is a UFD, prove that R is normal. Thus $\mathbb{Z}[x_1, \dots, x_n]$ and $k[x_1, \dots, x_n]$ are normal.
Hint: Suppose $f/g \in K(R)$ is integral over R where f and g have no common factor.

Solution: Suppose that $f/g \in K(R)$ is integral and f and g have no common factor. So there exist $a_{n-1}, \dots, a_0 \in R$ such that

$$(f/g)^n + a_{n-1}(f/g)^{n-1} + \dots + a_0 = 0.$$

Multiplying through by g^n gives us

$$f^n + a_{n-1}f^{n-1}g^1 + \dots + a_0 g^n = 0.$$

Thus $f^n \in \langle g \rangle$. Since f and g have no common factors and R is a UFD, the only way this can happen is if $\langle g \rangle = R$ and thus g is a unit in R. Thus $f/g \in K(R)$ and the problem is completed.

3. Suppose that R is a 1-dimensional local domain with maximal ideal $\mathfrak{m} = \langle r \rangle$. Prove that R is a UFD and thus normal.
Hint: Show all non-zero elements of R are of the form ur^n for some unit u.

Solution: Suppose that $x \in R$ and x is not of the form ur^n for some unit u. If x is a unit, then $x = ur^0$ and we have a contradiction. So $x \in \mathfrak{m}$. Write $x = x_1 r$, note x_1 can't be a unit, so $x = x_2 r^2$. Continuing in this way we see that $x \in \bigcap_{n \geq 0} \langle r^n \rangle =: J$. Now, consider $rJ = \mathfrak{m}J \subseteq J$. Take $y \in J$, so for each $n > 0$, $y = y_n r^n$ for some $y_n \in R$. Consider $z = y/r = y_n r^{n-1}$. This also has the prescribed form so $z \in J$ as well thus $y \in rJ$. Therefore $rJ = J$ and Nakayama's Lemma then implies that $J = 0$. This implies that all non-zero elements of R are of the form ur^n.

But this clearly implies that R is a UFD (since R is a domain).

We also have the following converse to **3.** which you may assume without proof (as needed).

Theorem 0.2. *If R is a 1-dimensional normal Noetherian ring, then* $\mathfrak{m} = \langle r \rangle$.

4. Suppose that R is an integral domain. Show that R is normal if each $R_{\mathfrak{q}}$ is normal for every prime ideal $\mathfrak{q} \in \operatorname{Spec} R$. *Hint:* You may use the fact that $R = \bigcap_{\mathfrak{q}} R_{\mathfrak{q}}$

Solution: Indeed, suppose that $x \in K(R)$ is integral over R. Since each $R_{\mathfrak{q}}$ is normal and contains R, x satisfies a monic polynomial with coefficients in $R_{\mathfrak{q}}$ as well. Thus $x \in R_{\mathfrak{q}}$ for all $\mathfrak{q}$. We are now done by the hint.

5. Prove the converse to **4.** In other words, suppose that R is an integral domain and R is normal. Prove that $R_{\mathfrak{q}}$ is also normal for all $\mathfrak{q} \in \operatorname{Spec} R$.

Solution: Suppose that z is in the normalization of $R_{\mathfrak{q}}$. Thus $z^n + (a_{n-1}/g_{n-1})z^{n-1} + \cdots + (a_1/g_1)z^1 + (a_0/g_0) = 0$ for some $a_i \in R$ and $g_i \in R \setminus \mathfrak{q} := W$. Multiplying through by $w^n = \prod_i g_i^n$, we obtain

$$(zw)^n + (a_{n-1}w/g_{n-1})(zw)^{n-1} + \cdots + (a_1 w^{n-1}/g_{n-1})(zw)^1 + (a_0 w^n/g_0) = 0$$

It is clear that each of the $a_{n-i}w^i/g_{n-i} \in R$ and so zw is integral over R and thus contained in R. But then $z = zw/w \in R_{\mathfrak{q}}$ as desired.

Theorem 0.3. *Suppose that S is either $\mathbb{Z}[x_1, \ldots, x_n]$ or $k[x_1, \ldots, x_n]$ and set $R = S/\langle f \rangle$ for some $f \in S$. In this case, R is normal if and only if $R_{\mathfrak{q}}$ is normal for every $\mathfrak{q} \in \operatorname{Spec} R$ such that $\dim R_{\mathfrak{q}} = 1$ (in other words, for every height-1 prime ideal)*

We will use this to construct an example of a normal ring which is not a UFD.

6. Assuming the theorem above (without proof), conclude that $R = k[x, y, z]/\langle xy - z^2 \rangle$ is a normal (it's not difficult to see that this is not a UFD since $xy = z^2$).
Hint: Consider the map $R \to k[s, t]$ which sends $x \mapsto s$, $y \mapsto st^2$ and $z \mapsto st$. Show that after inverting x and s, one gets an isomorphism. What height one primes are left?

Solution: So this is a little trickier, made harder still since I gave the wrong map in the hint. Set $S = k[s, t]$ with the given $\varphi : R \to S$. Note that $R[x^{-1}] = k[x, x^{-1}, y, z]/\langle y - z^2/x \rangle$ which clearly maps bijectively onto $k[s, s^{-1}, t]$. Of course, the height one primes of $R[x^{-1}]$ coincide with the height one primes of R that do not contain x. Since S is normal, so is $S[s^{-1}]$ and thus $S[s^{-1}]_{\mathfrak{q}}$ is normal for any height one prime. It then follows that $R_{\mathfrak{q}}$ is normal for any height one prime not containing x.

Likewise we can consider the map $\psi : R \to S$ defined by the rule $x \mapsto (st)^2$, $y \mapsto s$, $z \mapsto st$. By symmetry, it follows that $R_{\mathfrak{q}}$ is normal for any height one prime not containing y.

Finally, we notice that there are no height-one primes containing both x and y since $\sqrt{\langle x, y \rangle} = \langle x, y, z \rangle$ is certainly height at least 2 (note $\langle 0 \rangle \subseteq \langle x, z \rangle \subseteq \langle x, y, z \rangle$.) It then follows that $R_{\mathfrak{q}}$ is normal for every height one prime and so by the theorem we have completed the proof.