NOTES ON HOMOLOGICAL ALGEBRA

MATH 538 FALL 2011

1. Tor

Fix a ring A and a module M. A *free resolution* of M is a set of free modules $F_i = R^{\oplus n_i}$, $i \in \mathbb{Z}_{\geq 0}$ and maps $F_i \to F_{i-1}$ (set $F_{-1} = 0$) as well as a map $\rho : F_0 \to M$ such that

$$\cdots \to F_3 \to F_2 \to F_1 \to F_0 \xrightarrow{\rho} M \to 0$$

is exact. The data:

$$\cdots \to F_3 \to F_2 \to F_1 \to F_0 \to 0$$

is denoted by $F_\bullet$ and is called a *free resolution of* M. With the map to M, the data is denoted by $F_\bullet \to M$.

We briefly recall why free resolutions exist for every module. Indeed, choose a set of generators for M and fix a map $\rho : F_0 \to M$ sending the basis elements of F_0 to the generators of M. Choose a free module F_1 surjecting onto the kernel Z_0 of ρ. We then have $F_1 \to F_0 \to M \to)$ exact. Choose F_2 surjecting onto the kernel of $F_1 \to F_0$. Continuing in this way proves that every module has a free resolution.

Given another module N, we can form a complex denoted by $(F_\bullet) \otimes_A N$

$$\ldots \xrightarrow{d_4} F_3 \otimes_A N \xrightarrow{d_3} F_2 \otimes_A N \xrightarrow{d_2} F_1 \otimes_A N \xrightarrow{d_1} F_0 \otimes_A N \xrightarrow{d_0} 0.$$

By a *complex*, we just mean that $\ker d_i \supseteq \operatorname{Image} d_{i+1}$.

Definition 1.1. Given two A-modules M and N, we define

$$\operatorname{Tor}_i^A(M, N) = \ker d_i / \operatorname{Image} d_{i+1}$$

If A is implicitly understood, we may only write $\operatorname{Tor}_i(M, N)$.

I list several facts which I won't prove (they aren't any harder than what we do below). You should prove (v) to yourself though!

Proposition 1.1. *With M, N and A as above, we have:*

- (i) $\operatorname{Tor}_i(M, N)$ *is independent of the choice of free resolution for N.*
- (ii) *If one takes a flat resolution instead of a free resolution (ie, only requiring that the F_i are flat), then the same formula still works.*
- (iii) $\operatorname{Tor}_i(M, N) \cong \operatorname{Tor}_i(N, M)$ *(ie, take a free resolution of N, etc.)*
- (iv) $\operatorname{Tor}_i(-, N)$ *is a functor. In particular, if $M \to M'$ is a map of modules, then there is an induced map*
$$\operatorname{Tor}_i(M, N) \to \operatorname{Tor}_i(M', N).$$
- (v) $\operatorname{Tor}_0(M, N) \cong M \otimes_A N$.
- (vi) *If M is flat, then* $\operatorname{Tor}_i(M, N) = 0$ *for all $i > 0$.*

I recall a couple lemmas.

Lemma 1.1 (The Snake Lemma)**.** *Suppose that we have a commutative diagram with exact rows*

$$\begin{array}{ccccccc} & & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & & \end{array}$$

Then we have an exact sequence:

$$\ker \alpha \to \ker \beta \to \ker \gamma \to \operatorname{coker} \alpha \to \operatorname{coker} \beta \to \operatorname{coker} \gamma$$

Proof. This is in the homework. □

As an exercise, try to prove the following (important) lemma.

Lemma 1.2 (The Five Lemma)**.** *Suppose we have a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_1}\downarrow \\ M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \end{array}$$

- *Suppose that f_2 and f_4 are surjective and f_5 is injective, then f_3 is surjective.*
- *Suppose that f_2 and f_4 are injective and f_1 is surjective, then f_3 is injective.*
- *In particular if f_2 and f_4 are isomorphisms, f_1 is surjective and f_5 is injective, then f_3 is an isomorphism.*

Proof. This is a standard diagram chase. Try it. □

We now discuss the horseshoe lemma.

Lemma 1.3 (The horseshoe lemma)**.** *Suppose that we are given a diagram*

$$\begin{array}{ccccccccc} & & \cdots & & & & \cdots & & \\ & & \downarrow & & & & \downarrow & & \\ & & F_2' & & & & F_2'' & & \\ & & \downarrow & & & & \downarrow & & \\ & & F_1' & & & & F_1'' & & \\ & & \downarrow & & & & \downarrow & & \\ & & F_0' & & & & F_0'' & & \\ & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

with the vertical columns free resolutions and the horizontal row exact. Then set $F_i = F_i' \oplus F_i''$. Then one can use the F_i to form a commutative diagram:

$$\begin{array}{ccccccccc} & & \cdots & & \cdots & & \cdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_2' & \longrightarrow & F_2 & \longrightarrow & F_2'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_1' & \longrightarrow & F_1 & \longrightarrow & F_1'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_0' & \longrightarrow & F_0 & \longrightarrow & F_0'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

where the rows are exact (and are the obvious inclusion and projection maps) and each column is a free resolution.

Proof. First define $F_0 \to M$ using the fact that F_0 is free (and making all the squares in the diagram commute). To see that $F_0 \to M$ is surjective, use the snake lemma. The kernels of $F_0' \to M'$, $F_0 \to M$ and

$F_0'' \to M$ also form a short exact sequence by the snake lemma. Use this sequence instead of M and apply the snake lemma again. By induction, this will complete the proof. $\square$

The main reason for introducing $\mathrm{Tor}_i(M,N)$ is because of the long exact sequence of Tor's.

Theorem 1.2. *Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of A-modules and N is any other A-module. Then there is a long exact sequence:*

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & \mathrm{Tor}_i(M',N) & \longrightarrow & \mathrm{Tor}_i(M,N) & \longrightarrow & \mathrm{Tor}_i(M'',N) & \longrightarrow & \dots \\
 & & \dots & & \dots & & \dots & & \\
\dots & \longrightarrow & \mathrm{Tor}_2(M',N) & \longrightarrow & \mathrm{Tor}_2(M,N) & \longrightarrow & \mathrm{Tor}_2(M'',N) & \longrightarrow & \\
 & \longrightarrow & \mathrm{Tor}_1(M',N) & \longrightarrow & \mathrm{Tor}_1(M,N) & \longrightarrow & \mathrm{Tor}_1(M'',N) & \longrightarrow & \\
 & \longrightarrow & M' \otimes N & \longrightarrow & M \otimes N & \longrightarrow & M'' \otimes N & \longrightarrow & 0
\end{array}$$

here $\mathrm{Tor}_j(M'',N)$ maps to $\mathrm{Tor}_{j-1}(M'',N)$.

Proof. Use the horseshoe lemma to construct a diagram with exact rows

$$\begin{array}{ccccccccc}
 & & \dots & & \dots & & \dots & & \\
 & & \downarrow{\scriptstyle d_3'} & & \downarrow{\scriptstyle d_3} & & \downarrow{\scriptstyle d_3''} & & \\
0 & \longrightarrow & F_2' \otimes N & \longrightarrow & F_2 \otimes N & \longrightarrow & F_2'' \otimes N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_2'} & & \downarrow{\scriptstyle d_2} & & \downarrow{\scriptstyle d_2''} & & \\
0 & \longrightarrow & F_1' \otimes N & \longrightarrow & F_1 \otimes N & \longrightarrow & F_1'' \otimes N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_1'} & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_1''} & & \\
0 & \longrightarrow & F_0' \otimes N & \longrightarrow & F_0 \otimes N & \longrightarrow & F_0'' \otimes N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_0'} & & \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_0''} & & \\
0 & & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Note that the higher rows are exact on the left since F_i'' is free (or since the rows were split exact before tensoring). Note we replaced the bottom row from the horseshoe lemma with zeroes.

It follows the fact that the columns are complexes that we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & \frac{F_i' \otimes N}{d_{i+1}'(F_{i+1}' \otimes N)} & \longrightarrow & \frac{F_i \otimes N}{d_{i+1}(F_{i+1} \otimes N)} & \longrightarrow & \frac{F_i'' \otimes N}{d_{i+1}''(F_{i+1}'' \otimes N)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\
0 & \longrightarrow & \ker(d_{i-1}') & \longrightarrow & \ker(d_{i-1}) & \longrightarrow & \ker(d_{i-1}'') & &
\end{array}$$

It is not difficult to see that the cokernel of a is $\mathrm{Tor}_i(M',N)$, the cokernel of b is $\mathrm{Tor}_i(M,N)$ and the cokernel of c is $\mathrm{Tor}_i(M'',N)$. But one can also check easily that the kernel of a is $\mathrm{Tor}_{i+1}(M',N)$, the kernel of b is $\mathrm{Tor}_{i+1}(M,N)$ and the kernel of c is $\mathrm{Tor}_{i+1}(M'',N)$. An application of the snake lemma completes the proof. $\square$

2. Ext

Suppose that $F_\bullet \to M$ is a free resolution of an A-module M. Fix another A-module N and apply the functor $\mathrm{Hom}_A(\cdot, N)$. This gives us a complex:

$$0 \xrightarrow{d_0} \mathrm{Hom}_A(F_0,N) \xrightarrow{d_1} \mathrm{Hom}_A(F_1,N) \xrightarrow{d_2} \mathrm{Hom}_A(F_2,N) \xrightarrow{d_3} \dots$$

noting that $\mathrm{Hom}_A(\cdot, N)$ reverses arrows.

Definition 2.1. We define $\mathrm{Ext}^i_A(M,N)$ to be $\ker d_{i+1} / \ker d_i$.

Proposition 2.1. *With M, N and A as above, we have:*

(i) $\operatorname{Ext}^i_A(M,N)$ *is independent of the choice of free resolution for M.*
(ii) $\operatorname{Ext}^i_A(-,N)$ *is a functor. In particular, if $M \to M'$ is a map of modules, then there is an induced map*
$$\operatorname{Ext}^i_A(M',N) \to \operatorname{Ext}^i_A(M,N).$$
(iii) $\operatorname{Ext}^0_A(M,N) \cong \operatorname{Hom}_A(M,N)$.

Just as before, we have long exact sequences for Ext.

Theorem 2.2 (Long exact sequences for Ext #1)**.** *Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of A-modules and N is any other A-module. Then there is a long exact sequence*

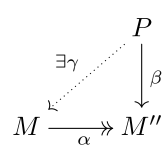
$$\begin{aligned} 0 &\longrightarrow \operatorname{Hom}_A(M'',N) \longrightarrow \operatorname{Hom}_A(M,N) \longrightarrow \operatorname{Hom}_A(M',N) \\ &\longrightarrow \operatorname{Ext}^1_A(M'',N) \longrightarrow \operatorname{Ext}^1_A(M,N) \longrightarrow \operatorname{Ext}^1_A(M',N) \\ &\longrightarrow \operatorname{Ext}^2_A(M'',N) \longrightarrow \operatorname{Ext}^2_A(M,N) \longrightarrow \operatorname{Ext}^2_A(M',N) \\ &\longrightarrow \operatorname{Ext}^3_A(M'',N) \longrightarrow \dots \end{aligned}$$

Proof. It is the same as the proof for Tor. □

We'd like to obtain for Ext a number of our other easy results for Tor. First we need a a little bit of framework.

Definition 2.3. An A-module P is called *projective* if any of the following equivalent conditions are satisfied:

- The functor $\operatorname{Hom}_A(P,\cdot)$ is an exact functor.
- If $M \to M''$ is surjective, then so is $\operatorname{Hom}_A(P,M) \to \operatorname{Hom}_A(P,M'')$.
- If $\alpha : M \to M''$ is surjective and $\beta : P \to M''$ is any map, then there exists a map γ making the diagram commute:



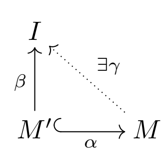
It is easy to see that free modules are projective.

Proposition 2.2. *If $F_\bullet \to M$ is a projective resolution of M (like a free resolution, but only require that the F_i are projective instead of free), then one still can compute $\operatorname{Ext}^i_A(M,N)$ as above using the projective resolution instead.*

There is a dual definition.

Definition 2.4. An A-module I is called *injective* if any of the following equivalent conditions are satisfied:

- The functor $\operatorname{Hom}_A(\cdot,I)$ is an exact functor.
- If $M' \to M$ is injective, then $\operatorname{Hom}_A(M,I) \to \operatorname{Hom}_A(M',I)$ is surjective.
- If $\alpha : M' \to M$ is injective and $\beta : M' \to I$ is any map, then there exists a map γ making the diagram commute:



Free modules are definitely not injective modules in general. However, injective modules are ubiquitous.

Proposition 2.3. *Given any module M, there exists an injective module I and an injection $M \hookrightarrow I$.*

The previous condition is called *having enough injectives*. It follows that for any module M, that there exists an exact sequence

$$0 \to M \to I_0 \to I_1 \to I_2 \to I_3 \to \dots$$

Indeed, start with $M \to I_0$ and set $K_0 = I_0/M$ to be the cokernel. Then find an injective module $I_1 \supseteq K_0$. This gives us

$$0 \to M \to I_0 \to I_1$$

exact. Now set I_2 to be an injective module containing $I_1/\operatorname{Image}(I_0)$ which yields $0 \to M \to I_0 \to I_1 \to I_2 \to \dots$ as desired. This is called an *injective resolution of* M and is denoted by $I_\bullet$ (without the map from M) or $M \to I_\bullet$.

Given modules M and N, take an injective resolution $N \to I_\bullet$ of N. Apply the functor $\operatorname{Hom}_A(M, \cdot)$ to $I_\bullet$. This gives us a complex

$$0 \xrightarrow{d_0} \operatorname{Hom}_A(M, I_0) \xrightarrow{d_1} \operatorname{Hom}_A(M, I_1) \xrightarrow{d_2} \operatorname{Hom}_A(M, I_2) \xrightarrow{d_3} \dots$$

Definition 2.5 (Ext redux). With notation as above, we define $\operatorname{Ext}^i_A(M, N)$ to be $\ker(d_{i+1})/\operatorname{Image}(d_i)$. This agrees with the previous definition.

Theorem 2.6 (Long exact sequence for Ext #2)**.** *Suppose that* $0 \to N' \to N \to N'' \to 0$ *is a short exact sequence of* A*-modules and* M *is any other* A*-module. Then there is a long exact sequence:*

$$\begin{aligned} 0 &\longrightarrow \operatorname{Hom}_A(M, N') \longrightarrow \operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_A(M, N'') \\ &\longrightarrow \operatorname{Ext}^1_A(M, N') \longrightarrow \operatorname{Ext}^1_A(M, N) \longrightarrow \operatorname{Ext}^1_A(M, N'') \\ &\longrightarrow \operatorname{Ext}^2_A(M, N') \longrightarrow \operatorname{Ext}^2_A(M, N) \longrightarrow \operatorname{Ext}^2_A(M, N'') \\ &\longrightarrow \operatorname{Ext}^3_A(M, N') \longrightarrow \dots \end{aligned}$$

Proof. The proof is again the same as before (except one uses injective modules instead of projectives/free modules). $\square$